University of Anbar

College of Engineering

Civil Engineering Department

LECTURE NOTE

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CALCULUS II

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CHAPTER ONE

EXPONENTIAL, LOGARITHMIC, AND INVERSE TRIGONOMETRIC FUNCTIONS

1.1 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

1.1.1 Irrational Exponents

- If b is a nonzero real number, then nonzero *integer* powers of b are defined by

$$b^n = b \times b \times \ldots \times b \text{ and } b^{-n} = \frac{1}{b^n}$$

- If p/q is a positive *rational* number expressed in lowest terms, then

$$b^{p/q} = \sqrt[q]{b^p} = \left(\sqrt[q]{b}\right)^p \text{ and } b^{-p/q} = \frac{1}{b^{p/q}}$$

- If b is negative, then some fractional powers of b will have imaginary values—the quantity $(-2)^{1/2} = \sqrt{-2}$, for example.
- There are various methods for defining irrational powers such as

$$2^{\pi}, 3^{\sqrt{2}}, \pi^{-\sqrt{5}}$$

Example: 2^{π}

x	2^x
3	8.000000
3.1	8.574188
3.14	8.815241
3.141	8.821353
3.1415	8.824411
3.14159	8.824962
3.141592	8.824974
3.1415926	8.824977

$$b^p b^q = b^{p+q}\ , \quad \frac{b^q}{b^q} = b^{p-q}, (b^p)^q = b^{pq}$$

1.1.2 The Family of Exponential Functions

- A function of the form $f(x) = b^x$, where $b > 0$, is called an ***exponential function with base b***. Some examples are

$$f(x) = 2^x, \quad f(x) = \left(\frac{1}{2}\right)^x, \quad f(x) = \pi^x$$

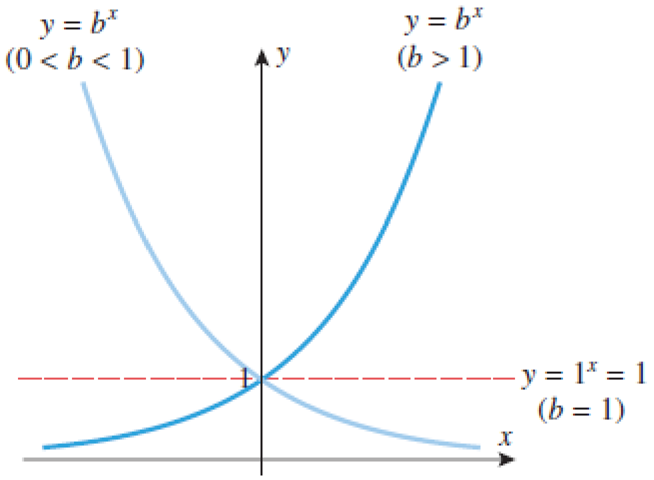


Figure 1-1

- The graph passes through (0, 1) because $b^0 = 1$.
- If $b > 1$, the value of b^x increases as x increases. As you traverse the graph of $y = b^x$ from left to right, the values of b^x increase indefinitely. If you traverse the graph from right to left, the values of b^x decrease toward zero but never reach zero. Thus, the x-axis is a horizontal asymptote of the graph of b^x.
- If $0 < b < 1$, the value of b^x decreases as x increases. As you traverse the graph of $y = b^x$ from left to right, the values of b^x decrease toward zero but never reach zero. Thus, the x-axis is a horizontal asymptote of the graph of b^x. If you traverse the graph from right to left, the values of b^x increase indefinitely.
- If $b = 1$, then the value of b^x is constant.

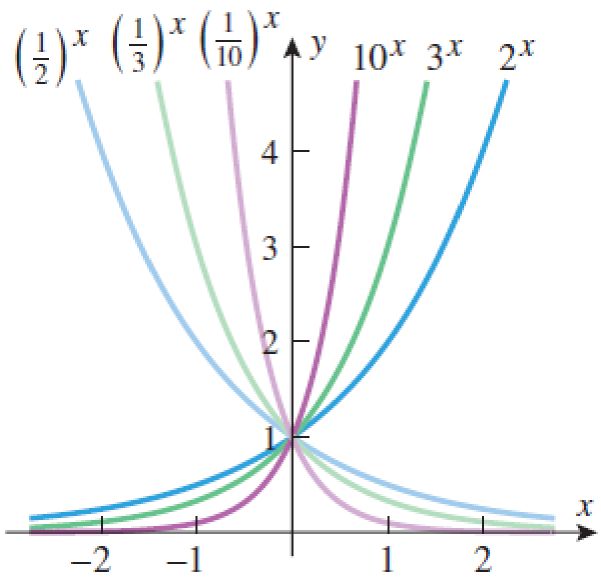


Figure 1-2 the family of b^x

- The graph of $y = (1/b)^x$ is the reflection of the graph of $y = b^x$ about the y-axis. This is because replacing x by $-x$ in the equation $y = b^x$ yields

$$y = b^{-x} = (1/b)^x$$

Theorem 1-1

If $b > 0$ and $b \neq 1$, then:

(a) *The function $f(x) = b^x$ is defined for all real values of x, so its natural domain is* $(-\infty, +\infty)$.

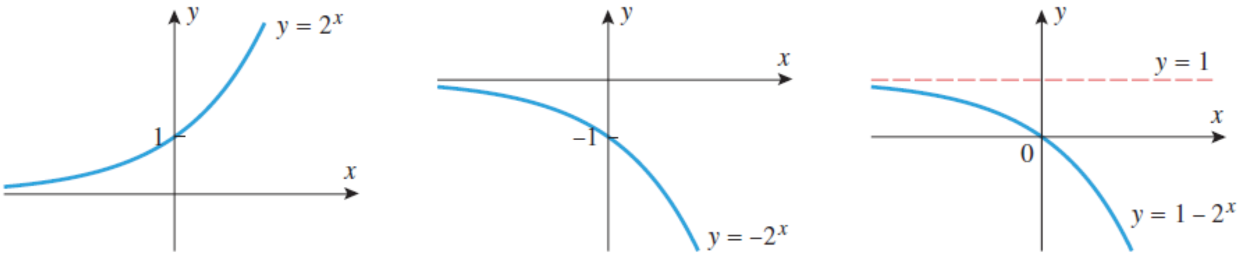
(b) *The function $f(x) = b^x$ is continuous on the interval* $(-\infty, +\infty)$, *and its range is* $(0, +\infty)$.

Example 1.1 Sketch the graph of the function $f(x) = 1 - 2^x$ and find its domain and range.

Solution:

1. $f(x) = 2^x$
2. $f(x) = -2^x$
3. $f(x) = 1-2^x$

The domain of f is $(-\infty, +\infty)$ and the range is $(-\infty, 1)$.

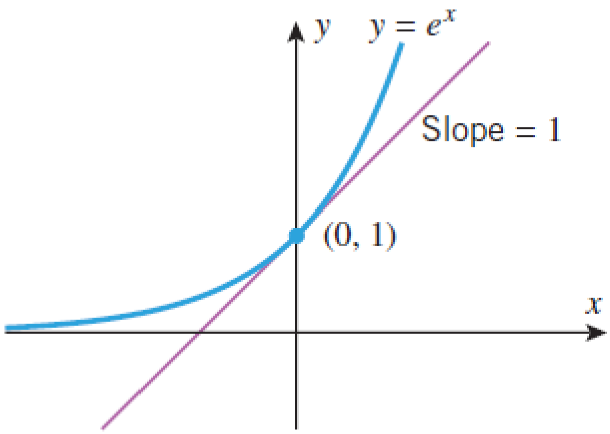


1.1.3 The Natural Exponential Function

Exponential functions (e), is a certain irrational number whose value to six decimal places is

$$e \approx 2.718282$$

- If $b = e$ and curve $y = b^x$ then $y = e^x$.
- The tangent line to the graph of $y = e^x$ at (0, 1) has slope 1.



The function $f(x) = e^x$ is called the ***natural exponential function***. To simplify typography, the natural exponential function is sometimes written as exp(x), in which case the relationship $e^{x1+x2} = e^{x1}e^{x2}$ would be expressed as

$$\exp(x_1 + x_2) = \exp(x_1)\exp(x_2)$$

- The constant e also arises in the context of the graph of the equation

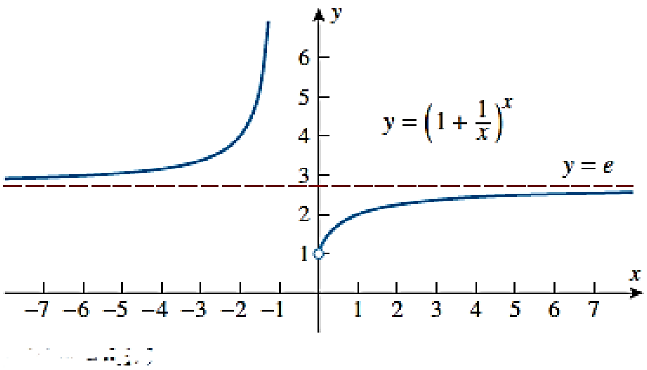
$$y = \left(1 + \frac{1}{x}\right)^x$$

- The below figure and table, $y = e$ is a horizontal asymptote of this graph, and the limits

$$\lim_{x\to+\infty}\left(1 + \frac{1}{x}\right)^x = e \quad and \quad \lim_{x\to-\infty}\left(1 + \frac{1}{x}\right)^x = e$$

- These limits can be derived from the limit

$$\lim_{x\to 0}(1 + x)^{1/x} = e$$



THE VALUES OF $(1 + 1/x)^x$ APPROACH e AS $x \to +\infty$

x	$1 + \frac{1}{x}$	$\left(1 + \frac{1}{x}\right)^x$
1	2	≈ 2.000000
10	1.1	2.593742
100	1.01	2.704814
1000	1.001	2.716924
10,000	1.0001	2.718146
100,000	1.00001	2.718268
1,000,000	1.000001	2.718280

Figure 1-3

1.1.4 Logarithmic Functions

- If $b > 0$ and $b \neq 1$, then for a positive value of x the expression

$$\log_b x \qquad \text{(read "the logarithm to the base } b \text{ of } x\text{")}$$

$$\log_{10} 100 = 2, \quad \log_{10}(1/1000) = -3, \quad \log_2 16 = 4, \quad \log_b 1 = 0, \quad \log_b b = 1$$

$$\boxed{10^2 = 100} \quad \boxed{10^{-3} = 1/1000} \quad \boxed{2^4 = 16} \quad \boxed{b^0 = 1} \quad \boxed{b^1 = b}$$

- We call the function $f(x) = \log_b x$ the ***logarithmic function with base b***.

Theorem 1.2

If $b > 0$ and $b \neq 1$, then b^x and $\log_b x$ are inverse functions.

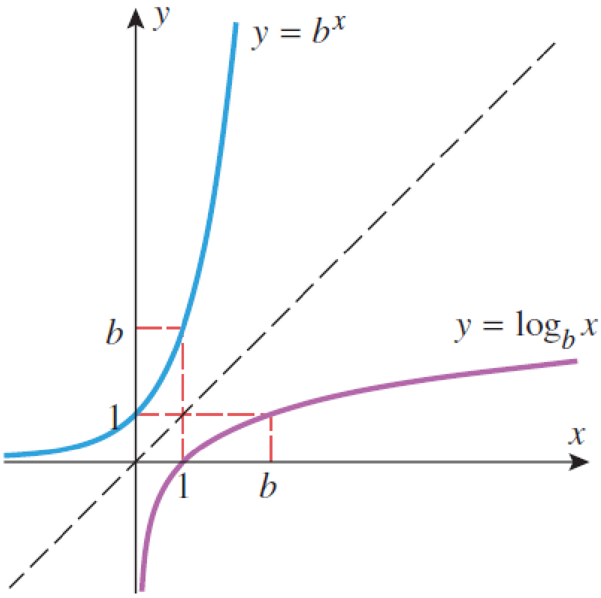


Figure 1-4

- The most important logarithm function which is the one with base e is called the ***natural logarithm function*** because the function $\log_e x$ is the inverse of the natural exponential function e^x. It is standard to denote the natural logarithm of x by $\ln x$ (read "ell en of x"), rather than loge x.

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$$\ln 1 = 0, \qquad \ln e = 1, \qquad \ln 1/e = -1, \qquad \ln(e^2) = 2$$

$\boxed{\text{Since } e^0 = 1}$ $\boxed{\text{Since } e^1 = e}$ $\boxed{\text{Since } e^{-1} = 1/e}$ $\boxed{\text{Since } e^2 = e^2}$

As shown in below table, the inverse relationship between b^x and $\log_b x$ produces a correspondence between some basic properties of those functions.

CORRESPONDENCE BETWEEN PROPERTIES OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

PROPERTY OF b^x	PROPERTY OF $\log_b x$
$b^0 = 1$	$\log_b 1 = 0$
$b^1 = b$	$\log_b b = 1$
Range is $(0, +\infty)$	Domain is $(0, +\infty)$
Domain is $(-\infty, +\infty)$	Range is $(-\infty, +\infty)$
x-axis is a horizontal asymptote	y-axis is a vertical asymptote

$$\begin{aligned} \log_b(b^x) &= x \quad \text{for all real values of } x \\ b^{\log_b x} &= x \quad \text{for } x > 0 \end{aligned}$$

- In special case where b=e, these equations become

$$\begin{aligned} \ln(e^x) &= x \quad \text{for all real values of } x \\ e^{\ln x} &= x \quad \text{for } x > 0 \end{aligned}$$

- In words, the functions b^x and $\log_b x$ cancel out the effect of one another when composed in either order; for example,

$$\log 10^x = x, \quad 10^{\log x} = x, \quad \ln e^x = x, \quad e^{\ln x} = x, \quad \ln e^5 = 5, \quad e^{\ln \pi} = \pi$$

1.1.5 Solving Equations Involving Exponentials and Logarithms

Theorem 1.3 (***Algebraic Properties of Logarithms***)

If $b > 0$, $b \neq 1$, $a > 0$, $c > 0$, *and* r *is any real number, then:*

(a) $\log_b (ac) = \log_b a + \log_b c$	Product property
(b) $\log_b (a/c) = \log_b a - \log_b c$	Quotient property
(c) $\log_b (a^r) = r \log_b a$	Power property
(d) $\log_b (1/c) = -\log_b c$	Reciprocal property

Example 1.2 Find x such that

(a) $\log x = \sqrt{2}$ (b) $\ln(x + 1) = 5$ (c) $5^x = 7$

Solution (a). Converting the equation to exponential form yields

$$x = 10^{\sqrt{2}} \approx 25.95$$

Solution (b). Converting the equation to exponential form yields

$$x + 1 = e^5 \text{ or } x = e^5 - 1 \approx 147.41$$

Solution (c). Converting the equation to logarithmic form yields

$$x = \log_5 7 \approx 1.21$$

Alternatively,

$$x \ln 5 = \ln 7 \quad \text{or} \quad x = \ln 7/\ln 5 \approx 1.21$$

Example 1.3 A satellite that requires 7 watts of power to operate at full capacity is equipped with a radioisotope power supply whose power output P in watts is given by the equation

$$P = 75\, e^{-t/125}$$

where t is the time in days that the supply is used. How long can the satellite operate at full capacity?

Solution. The power P will fall to 7 watts when

$$7 = 75e^{-t/125}$$

The solution for t is as follows:

$$7/75 = e^{-t/125}$$

$$\ln(7/75) = \ln(e^{-t/125})$$

$$\ln(7/75) = -t/125$$

$$t = -125 \ln(7/75) \approx 296.4$$

so the satellite can operate at full capacity for about 296 days.

Example 1.4 Solve $\frac{e^x - e^{-x}}{2} = 1$ for x

Solution. Multiplying both sides of the given equation by 2 yields

$$e^x - e^{-x} = 2$$

or equivalently,

$$e^x - (1/e^x) = 2$$

Multiplying through by e^x yields

$$e^{2x} - 1 = 2e^x \qquad \text{or} \qquad e^{2x} - 2e^x - 1 = 0$$

This is really a quadratic equation in disguise, as can be seen by rewriting it in the form

$$(e^x)^2 - 2e^x - 1 = 0$$

and letting $u = e^x$ to obtain

$$u^2 - 2u - 1 = 0$$

Solving for u by the quadratic formula yields

$$u = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

Since $u = e^x$

$$e^x = 1 \pm \sqrt{2}$$

But e^x cannot be negative

$$\begin{aligned} e^x &= 1 + \sqrt{2} \\ \ln e^x &= \ln(1 + \sqrt{2}) \\ x &= \ln(1 + \sqrt{2}) \approx 0.881 \end{aligned}$$

1.1.6 Change of Base Formula for Logarithms

$$\log_b x = \frac{\ln x}{\ln b}$$

Example 1.5 Use a calculating utility to evaluate $\log_2 5$ by expressing this logarithm in terms of natural logarithms.

Solution.

$$\log_2 5 = \ln 5 / \ln 2 \approx 2.321928$$

1.1.7 Logarithmic Scales in Science and Engineering

- Logarithms are used in science and engineering to deal with quantities whose units vary over an excessively wide range of values.
- For example, the "loudness" of a sound can be measured by its ***intensity*** I (in watts per square meter), which is related to the energy transmitted by the sound wave—the greater the intensity, the greater the transmitted energy, and the louder the sound is perceived by the human ear.
- ***Sound level*** β,

$$\beta = 10 \log(I / I_0)$$

where $I_0 = 10^{-12}$ W/m^2 is a reference intensity close to the threshold of human hearing, I is intensity.

Example 1.6 A space shuttle taking off generates a sound level of 150 dB near the launch-pad. A person exposed to this level of sound would experience severe physical injury. By comparison, a car horn at one meter has a sound level of 110 dB, near the threshold of pain for many people. What is the ratio of sound intensity of a space shuttle take off to that of a car horn?

Solution. Let I_1 and β_1 (= 150 dB) denote the sound intensity and sound level of the space shuttle taking off, and let I_2 and β_2 (= 110 dB) denote the sound intensity and sound level of a car horn. Then

$$I_1 / I_2 = (I_1 / I_0) / (I_2 / I_0)$$

$$\log (I_1 / I_2) = \log (I1 / I_0) - \log (I_2 / I_0)$$

$$10 \log (I_1 / I_2) = 10 \log (I_1 / I_0) - 10 \log (I_2 / I_0) = \beta_1 - \beta_2$$

$$10 \log (I_1 / I_2) = 150 - 100 = 40$$

$$\log (I_1 / I_2) = 4$$

Thus, $I_1 / I_2 = 104$, which tells us that the sound intensity of the space shuttle taking off is 10,000 times greater than a car horn!

1.2 DERIVATIVES AND INTEGRALS INVOLVING LOGARITHMIC FUNCTIONS

1.2.1 Derivatives of Logarithmic Functions

- $f(x) = \ln x$ is differentiable for $x > 0$ by using the derivative definition to find its derivative. To obtain this derivative, we need the fact that $\ln x$ is continuous for $x > 0$.
- Since e^x is continuous, we know that $\ln x$ is continuous for $x > 0$.

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$$

A derivative formula for the general logarithmic function $\log_b x$ can be obtained:

$$\frac{d}{dx}[\log_b x] = \frac{d}{dx}\left[\frac{\ln x}{\ln b}\right] = \frac{1}{\ln b}\frac{d}{dx}[\ln x]$$

$$\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}, \quad x > 0$$

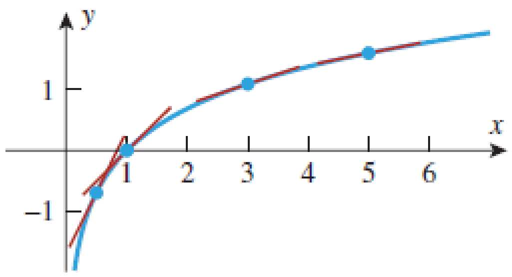
Example 1.7

(a) Figure 1.5 shows the graph of $y = \ln x$ and its tangent lines at the points $x = 1/2$, 1, 3, and 5. Find the slopes of those tangent lines.

(b) Does the graph of $y = \ln x$ have any horizontal tangent lines? Use the derivative of $\ln x$ to justify your answer.

Solution (a). The slopes of the tangent lines at the points $x = 1/2$, 1, 3, and 5 are $1/x = 2$, 1, 1/3, and 1/5, respectively, which is consistent with Figure 1.5.

Solution (b). It does not appear from the graph of $y = \ln x$ that there are any horizontal tangent lines. This is confirmed by the fact that $dy/dx = 1/x$ is not equal to zero for any real value of x.



$y = \ln x$ with tangent lines

Figure 1-5

If u is a differentiable function of x, and if $u(x) > 0$, then applying the chain rule to produce the following generalized derivative formulas:

$$\frac{d}{dx}[\ln u] = \frac{1}{u}\cdot\frac{du}{dx} \quad \text{and} \quad \frac{d}{dx}[\log_b u] = \frac{1}{u\ln b}\cdot\frac{du}{dx}$$

Example 1.8: Find

$$\frac{d}{dx}[\ln(x^2+1)].$$

Solution. $u = x^2 + 1$ we obtain

$$\frac{d}{dx}[\ln(x^2+1)] = \frac{1}{x^2+1}\cdot\frac{d}{dx}[x^2+1] = \frac{1}{x^2+1}\cdot 2x = \frac{2x}{x^2+1}$$

Example 1.9: Find

$$\begin{aligned}\frac{d}{dx}\left[\ln\left(\frac{x^2\sin x}{\sqrt{1+x}}\right)\right] &= \frac{d}{dx}\left[2\ln x + \ln(\sin x) - \frac{1}{2}\ln(1+x)\right]\\ &= \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{1}{2(1+x)}\\ &= \frac{2}{x} + \cot x - \frac{1}{2+2x}\end{aligned}$$

The derivative of $\ln |x|$ for $x \neq 0$ can be obtained by considering the cases $x > 0$ and $x < 0$ separately:

Case $x > 0$. In this case $|x| = x$, so

$$\frac{d}{dx}[\ln|x|] = \frac{d}{dx}[\ln x] = \frac{1}{x}$$

Case $x < 0$. In this case $|x| = -x$, so it follows that

$$\frac{d}{dx}[\ln|x|] = \frac{d}{dx}[\ln(-x)] = \frac{1}{(-x)}\cdot\frac{d}{dx}[-x] = \frac{1}{x}$$

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x} \quad \text{if } x \neq 0$$

Example 1.10

$$\frac{d}{dx}[\ln|\sin x|] = \frac{1}{\sin x} \cdot \frac{d}{dx}[\sin x] = \frac{\cos x}{\sin x} = \cot x$$

1.2.2 Logarithmic Differentiation

Example 1.11 The derivative of

$$y = \frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4}$$

Solution:

$$\ln y = 2\ln x + \tfrac{1}{3}\ln(7x-14) - 4\ln(1+x^2)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{7/3}{7x-14} - \frac{8x}{1+x^2}$$

$$\frac{dy}{dx} = \frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4}\left[\frac{2}{x} + \frac{1}{3x-6} - \frac{8x}{1+x^2}\right]$$

1.2.3 Integrals Involving ln *x*

The function $\ln x$ is an antiderivative of $1/x$ on the interval $(0, +\infty)$, whereas the function $\ln|x|$ is an antiderivative of $1/x$ on each of the intervals $(-\infty, 0)$ and $(0, +\infty)$.

$$\int \frac{1}{u}\,du = \ln|u| + C$$

Example 1.12 Evaluate

$$\int \frac{3x^2}{x^3+5}\,dx$$

Solution:

$$u = x^3 + 5, \quad du = 3x^2\,dx$$

$$\int \frac{3x^2}{x^3+5}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|x^3+5| + C$$

Example 1.13: Evaluate

$$\int \boldsymbol{tanx\ dx}$$

Solution:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{1}{u}\, du = -\ln|u| + C = -\ln|\cos x| + C$$

Important point: any integral of the form

$$\int \frac{g'(x)}{g(x)}\, dx$$

(where the numerator of the integrand is the derivative of the denominator) can be evaluated by the u-substitution $u = g(x)$, $du = g'(x)\, dx$, since this substitution yields

$$\int \frac{g'(x)}{g(x)}\, dx = \int \frac{du}{u} = \ln|u| + C = \ln|g(x)| + C$$

1.2.4 Derivatives of Real Powers of *x*

$$\frac{d}{dx}[x^r] = rx^{r-1}$$

Let $y = x^r$, where r is a real number. The derivative dy/dx can be obtained by logarithmic differentiation as follows:

$$\ln|y| = \ln|x^r| = r\ln|x|$$

$$\frac{d}{dx}[\ln|y|] = \frac{d}{dx}[r\ln|x|]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{r}{x}$$

$$\frac{dy}{dx} = \frac{r}{x}y = \frac{r}{x}x^r = rx^{r-1}$$

1.3 DERIVATIVES OF INVERSE FUNCTIONS; DERIVATIVES AND INTEGRALS INVOLVING EXPONENTIAL FUNCTIONS

1.3.1 Differentiability of Inverse Functions

Example 1.14: Suppose that f is a one-to-one differentiable function such that $f(2) = 1$ and $f'(2) = 3/4$. Then the tangent line to $y = f(x)$ at the point (2, 1) has equation

$$y - 1 = \frac{3}{4}(x - 2)$$

Since the graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about the line $y = x$, the tangent line to $y = f^{-1}(x)$ at the point (1, 2) is the reflection about the line $y = x$ of the tangent line to $y = f(x)$ at the point (2, 1) (Figure 1-6). Its equation can be obtained from that of the tangent line to the graph of f by interchanging x and y:

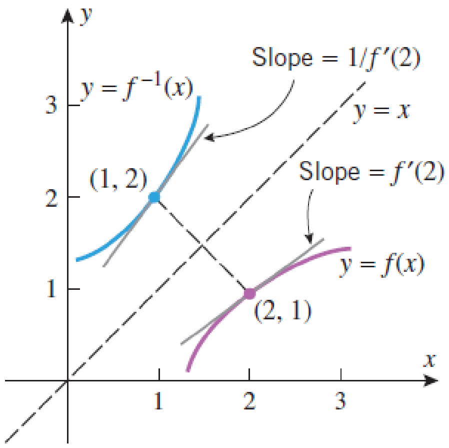


Figure 1-6

$x - 1 = 3/4\,(y - 2)$ or $y - 2 = 4/3\,(x - 1)$

Notice that the slope of the tangent line to $y = f^{-1}(x)$ at $x = 1$ is the reciprocal of the slope of the tangent line to $y = f(x)$ at $x = 2$. That is,

$$(f^{-1})'(1) = 1/f'(2) = 4/3 \qquad (1)$$

Since $2 = f^{-1}(1)$ for the function f in this example, it follows that $f'(2) = f'(f^{-1}(1))$. Thus, Formula (1) can also be expressed as

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$$

In general, if f is a differentiable and one-to-one function, then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \qquad (2)$$

provided $f'(f^{-1}(x)) \neq 0$.

Formula (2) can be confirmed using implicit differentiation. The equation $y = f^{-1}(x)$ is equivalent to $x = f(y)$. Differentiating with respect to x we obtain

$$1 = \frac{d}{dx}[x] = \frac{d}{dx}[f(y)] = f'(y) \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

Also from $x = f(y)$ we have $dx/dy = f'(y)$, which gives the following alternative version of Formula (2):

$$\frac{dy}{dx} = \frac{1}{dx/dy} \qquad (3)$$

1.3.2 Increasing or Decreasing Functions are One-To-One

Theorem 1.4

Suppose that the domain of a function f is an open interval on which $f'(x) > 0$ or on which $f'(x) < 0$. Then f is one-to-one, $f^{-1}(x)$ is differentiable at all values of x in the range of f, and the derivative of $f^{-1}(x)$ is given by Formula (2).

Example 1.15: Consider the function $f(x) = x^5 + x + 1$.

(a) Show that f has a differentiable inverse function.

(b) Use implicit differentiation to find a formula for the derivative of f^{-1}.

(c) Compute $(f^{-1})'(1)$.

Solution (a). Since $f'(x) = 5x^4 + 1 > 0$ for all real values of x, it follows from Theorem 1.4 that f is one-to-one on the interval $(-\infty, +\infty)$ and has a differentiable inverse function.

Solution (b). Let $y = f^{-1}(x)$. Differentiating $x = f(y) = y^5 + y + 1$ implicitly with respect to x yields

$$\frac{d}{dx}[x] = \frac{d}{dx}[y^5 + y + 1]$$

$$1 = (5y^4 + 1)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{5y^4 + 1} \qquad (4)$$

Solution (c). From Equation (4),

$$(f^{-1})'(1) = \frac{dy}{dx}\bigg|_{x=1} = \frac{1}{5y^4+1}\bigg|_{x=1}$$

Thus, we need to know the value of $y = f^{-1}(x)$ at $x = 1$, which we can obtain by solving the equation $f(y) = 1$ for y. This equation is $y^5 + y + 1 = 1$, which, by inspection, is satisfied by $y = 0$. Thus,

$$(f^{-1})'(1) = \frac{1}{5y^4+1}\bigg|_{y=0} = 1$$

1.3.3 Derivatives of Exponential Functions

$$\frac{d}{dx}[b^x] = b^x \ln b \qquad (5)$$

$$\frac{d}{dx}[e^x] = e^x \qquad (6)$$

If u is a differentiable function of x, then it follows from (5) and (6) that

$$\frac{d}{dx}[b^u] = b^u \ln b \cdot \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx} \qquad (7\ \&\ 8)$$

Example 1.16: The following computations use Formulas (5), (7) and (8).

$$\frac{d}{dx}[2^x] = 2^x \ln 2$$

$$\frac{d}{dx}[e^{-2x}] = e^{-2x} \cdot \frac{d}{dx}[-2x] = -2e^{-2x}$$

$$\frac{d}{dx}[e^{x^3}] = e^{x^3} \cdot \frac{d}{dx}[x^3] = 3x^2 e^{x^3}$$

$$\frac{d}{dx}[e^{\cos x}] = e^{\cos x} \cdot \frac{d}{dx}[\cos x] = -(\sin x)e^{\cos x}$$

Example 1.16: Use logarithmic differentiation to find d/dx $[(x^2 + 1)^{\sin x}]$

Solution. Setting $y = (x^2 + 1)^{\sin x}$ we have

$$\ln y = \ln[(x^2 + 1)^{\sin x}] = (\sin x)\ln(x^2 + 1)$$

Differentiating both sides with respect to x yields

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}[(\sin x)\ln(x^2+1)]\\ &= (\sin x)\frac{1}{x^2+1}(2x) + (\cos x)\ln(x^2+1)\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= y\left[\frac{2x\sin x}{x^2+1} + (\cos x)\ln(x^2+1)\right]\\ &= (x^2+1)^{\sin x}\left[\frac{2x\sin x}{x^2+1} + (\cos x)\ln(x^2+1)\right]\end{aligned}$$

1.3.4 Integrals Involving Exponential Functions

$$\int b^u\,du = \frac{b^u}{\ln b} + C \quad \text{and} \quad \int e^u\,du = e^u + C \tag{9 \& 10}$$

Example 1.17:

$$\int 2^x\,dx = \frac{2^x}{\ln 2} + C$$

Example 1.18:

$$\text{Evaluate} \int e^{5x}\,dx.$$

Solution. Let $u = 5x$ so that $du = 5\,dx$ or $dx = 1/5\,du$, which yields

$$\int e^{5x}\,dx = \frac{1}{5}\int e^u\,du = \frac{1}{5}e^u + C = \frac{1}{5}e^{5x} + C$$

Example 1.19: The following computations use Formula (10).

Solution.

$$\int e^{-x}\,dx = -\int e^{u}\,du = -e^{u} + C = -e^{-x} + C$$

$$\boxed{\begin{array}{l} u = -x \\ du = -dx \end{array}}$$

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^{u}\,du = \frac{1}{3}e^{u} + C = \frac{1}{3}e^{x^3} + C$$

$$\boxed{\begin{array}{l} u = x^3 \\ du = 3x^2\,dx \end{array}}$$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^{u}\,du = 2e^{u} + C = 2e^{\sqrt{x}} + C$$

$$\boxed{\begin{array}{l} u = \sqrt{x} \\ du = \dfrac{1}{2\sqrt{x}}\,dx \end{array}}$$

Example 1.19: Evaluate

$$\int_0^{\ln 3} e^{x}(1+e^{x})^{1/2}\,dx$$

Solution. Make the u-substitution

$$u = 1 + e^{x},\ du = e^{x}\,dx$$

and change the x-limits of integration ($x = 0$, $x = \ln 3$) to the u-limits

$$u = 1 + e^{0} = 2,\ u = 1 + e^{\ln 3} = 1 + 3 = 4$$

This yields

$$\int_0^{\ln 3} e^{x}(1+e^{x})^{1/2}\,dx = \int_2^4 u^{1/2}\,du = \frac{2}{3}u^{3/2}\Big]_2^4 = \frac{2}{3}[4^{3/2} - 2^{3/2}] = \frac{16 - 4\sqrt{2}}{3}$$

1.4 GRAPHS AND APPLICATIONS INVOLVING LOGARITHMIC AND EXPONENTIAL FUNCTIONS

1.4.1 Some Properties of e^x and ln *x*

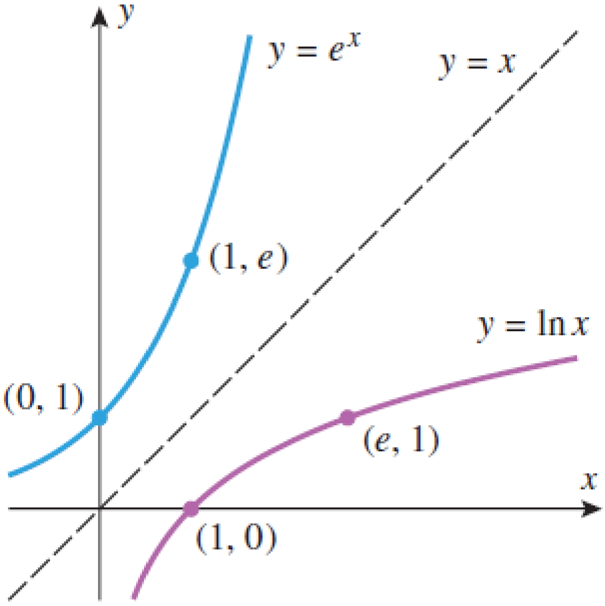


Figure 1-7

PROPERTIES OF e^x	PROPERTIES OF $\ln x$
$e^x > 0$ for all x	$\ln x > 0$ if $x > 1$ $\ln x < 0$ if $0 < x < 1$ $\ln x = 0$ if $x = 1$
e^x is increasing on $(-\infty, +\infty)$	$\ln x$ is increasing on $(0, +\infty)$
The graph of e^x is concave up on $(-\infty, +\infty)$	The graph of $\ln x$ is concave down on $(0, +\infty)$

$$\frac{d}{dx}[e^x] = e^x > 0 \quad \text{and} \quad \frac{d^2}{dx^2}[e^x] = \frac{d}{dx}[e^x] = e^x > 0$$

We can verify that $y = e^x$ is increasing and its graph is concave up from its first and second derivatives. For all x in $(-\infty, +\infty)$ we have

The first of these inequalities demonstrates that e^x is increasing on $(-\infty, +\infty)$, and the second inequality shows that the graph of $y = e^x$ is concave up on $(-\infty, +\infty)$. Similarly, for all x in $(0, +\infty)$ we have

$$\frac{d}{dx}[\ln x] = \frac{1}{x} > 0 \quad \text{and} \quad \frac{d^2}{dx^2}[\ln x] = \frac{d}{dx}\left[\frac{1}{x}\right] = -\frac{1}{x^2} < 0$$

1.4.2 Graphing Exponential and Logarithmic Functions

Example 1.20: Sketch the graph of $y = e^{-x^2/2}$ and identify the locations of all relative extrema and inflection points.

Solution:

• *Symmetries:* Replacing x by $-x$ does not change the equation, so the graph is symmetric about the y-axis.

• *x- and y-intercepts:* Setting $y = 0$ leads to the equation $e^{-x^2/2} = 0$, which has no solutions since all powers of e have positive values. Thus, there are no x-intercepts.
Setting $x = 0$ yields the y-intercept $y = 1$.

• *Vertical asymptotes:* There are no vertical asymptotes since $e^{-x^2/2}$ is continuous on $(-\infty, +\infty)$.

• *End behaviour:* The x-axis ($y = 0$) is a horizontal asymptote since

$$\lim_{x\to-\infty} e^{-x^2/2} = \lim_{x\to+\infty} e^{-x^2/2} = 0$$

• *Derivatives:*

$$\frac{dy}{dx} = e^{-x^2/2}\frac{d}{dx}\left[-\frac{x^2}{2}\right] = -xe^{-x^2/2}$$

$$\frac{d^2y}{dx^2} = -x\frac{d}{dx}[e^{-x^2/2}] + e^{-x^2/2}\frac{d}{dx}[-x]$$

$$= x^2e^{-x^2/2} - e^{-x^2/2} = (x^2 - 1)e^{-x^2/2}$$

Conclusions and graph:

• The sign analysis of y in Figure 1-8a is based on the fact that $e^{-x^2/2} > 0$ for all x.
This shows that the graph is always above the x-axis.

• The sign analysis of dy/dx in Figure 1-8a is based on the fact that $dy/dx = -xe^{-x^2/2}$ has the same sign as $-x$. This analysis and the first derivative test show that there is a stationary point at $x = 0$ at which there is a relative maximum. The value of y at the relative maximum is $y = e^0 = 1$.

• The sign analysis of d^2y/dx^2 in Figure 1-8*a* is based on the fact that $d^2y/dx^2 = (x^2 - 1)e^{-x^2/2}$ has the same sign as $x^2 - 1$. This analysis shows that there are inflection points at $x = -1$ and $x = 1$. The graph changes from concave up to concave down at $x = -1$ and from concave down to concave up at $x = 1$. The coordinates of the inflection points are $(-1, e^{-1/2}) \approx (-1, 0.61)$ and $(1, e^{-1/2}) \approx (1, 0.61)$.

The graph is shown in Figure 1-8*b*.

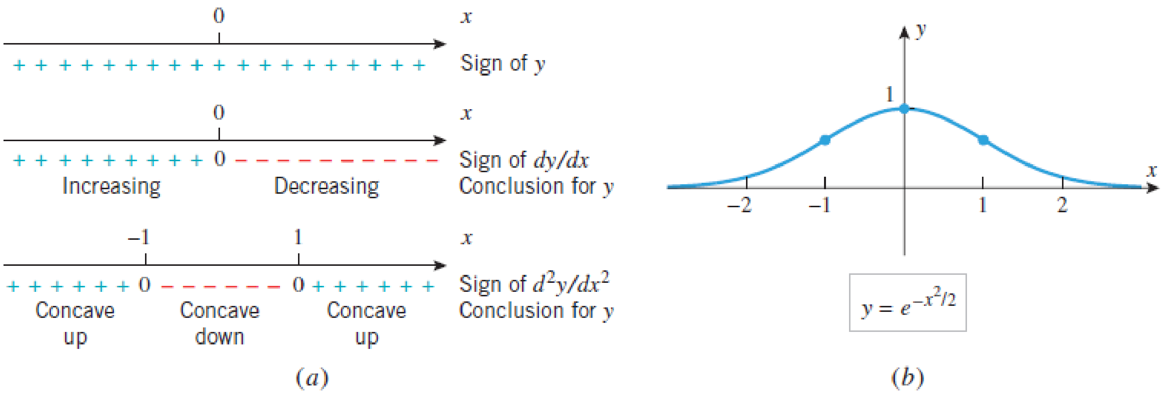


Figure 1-8

Example 1.21: Use a graphing utility to generate the graph of $f(x) = (\ln x)/x$, and discuss what it tells you about relative extrema, inflection points, asymptotes, and end behaviour. Use calculus to find the locations of all key features of the graph.

Solution. Figure 1-9 shows a graph of f produced by a graphing utility. The graph suggests that there is an x-intercept near $x = 1$, a relative maximum somewhere between $x = 0$ and $x = 5$, an inflection point near $x = 5$, a vertical asymptote at $x = 0$, and possibly a horizontal asymptote $y = 0$. For a more precise analysis of this information we need to consider the derivatives

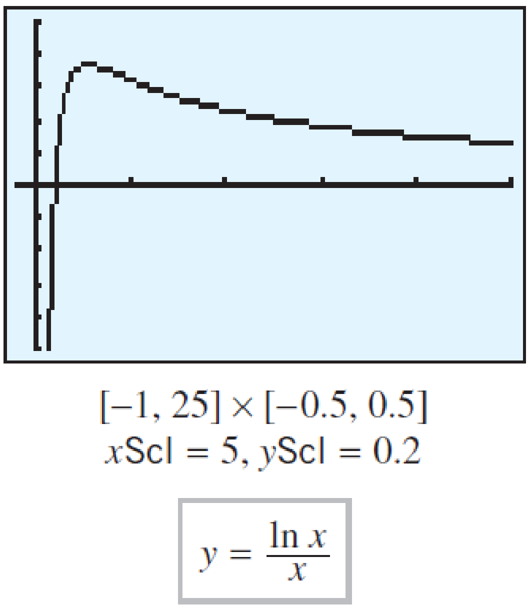


Figure 1-9

$$f'(x) = \frac{x\left(\frac{1}{x}\right) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

$$f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x)(2x)}{x^4} = \frac{2x\ln x - 3x}{x^4} = \frac{2\ln x - 3}{x^3}$$

Relative extrema: Solving $f'(x) = 0$ yields the stationary point $x = e$ (verify). Since

$$f''(e) = \frac{2-3}{e^3} = -\frac{1}{e^3} < 0$$

there is a relative maximum at $x = e \approx 2.7$ by the second derivative test.

• *Inflection points:* Since $f(x) = (\ln x)/x$ is only defined for positive values of x, the second derivative $f''(x)$ has the same sign as $2 \ln x - 3$. We leave it for you to use the inequalities $(2 \ln x - 3) < 0$ and $(2 \ln x - 3) > 0$ to show that $f''(x) < 0$ if $x < e^{3/2}$ and $f''(x) > 0$ if $x > e^{3/2}$. Thus, there is an inflection point at $x = e^{3/2} \approx 4.5$.

• *Asymptotes:* We will allow us to conclude that

$$\lim_{x \to +\infty} \frac{\ln x}{x} = 0$$

so that $y = 0$ is a horizontal asymptote. Also, there is a vertical asymptote at $x = 0$ since

$$\lim_{x \to 0^+} \frac{\ln x}{x} = -\infty$$

Intercepts: Setting $f(x) = 0$ yields $(\ln x)/x = 0$. The only real solution of this equation is $x = 1$, so there is an x-intercept at this point.

1.4.3 Logistic Curves

- When a population grows in an environment in which space or food is limited, the graph of population versus time is typically an S-shaped curve of the form shown in Figure 1-10.
- The scenario described by this curve is a population that grows slowly at first and then more and more rapidly as the number of individuals producing offspring increases. However, at a certain point in time (where the inflection point occurs) the environmental factors begin to show their effect, and the growth rate begins a steady decline.
- Over an extended period of time the population approaches a limiting value that represents the upper limit on the number of individuals that the available space or

food can sustain. Population growth curves of this type are called ***logistic growth curves***.

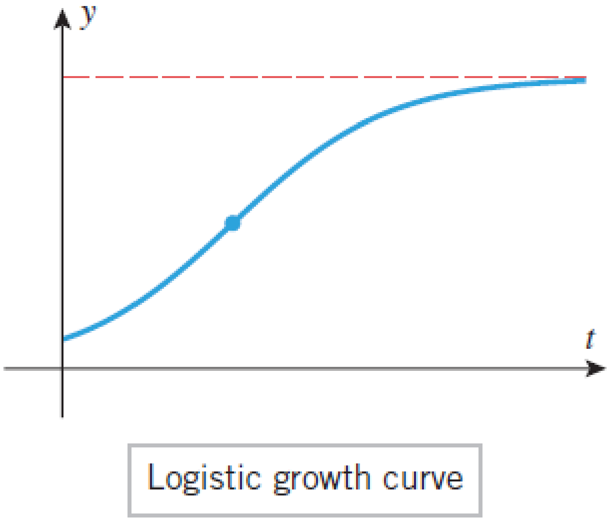


Figure 1-10

Example 1.22 We will see in a later chapter that logistic growth curves arise from equations of the form

$$y = \frac{L}{1 + Ae^{-kt}}$$

where y is the population at time t ($t \geq 0$) and A, k, and L are positive constants. Show that Figure 1-11 correctly describes the graph of this equation when $A > 1$.

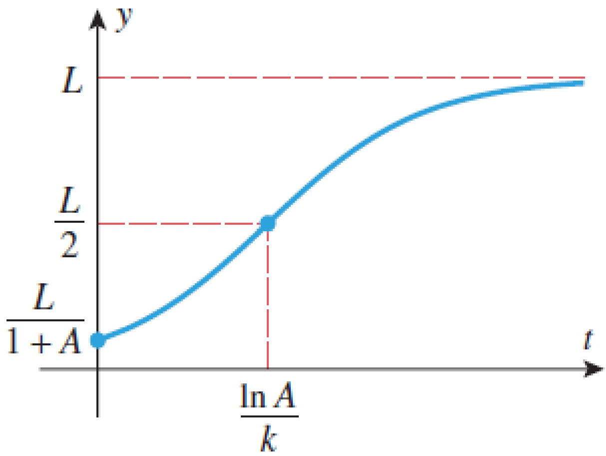


Figure 1-11

1.4.4 Newton's Law of Cooling

Example 1.23 A glass of lemonade with a temperature of 40◦F is left to sit in a room whose temperature is a constant 70◦F. Using a principle of physics called ***Newton's Law of Cooling***, one can show that if the temperature of the lemonade reaches 52◦F in 1 hour, then the temperature T of the lemonade as a function of the elapsed time t is modelled by the equation

$$T = 70 - 30e^{-0.5t}$$

where T is in degrees Fahrenheit and t is in hours. The graph of this equation, shown in Figure 1-12, conforms to our everyday experience that the temperature of the lemonade gradually approaches the temperature of the room. Find the average temperature T_{ave} of the lemonade over the first 5 hours.

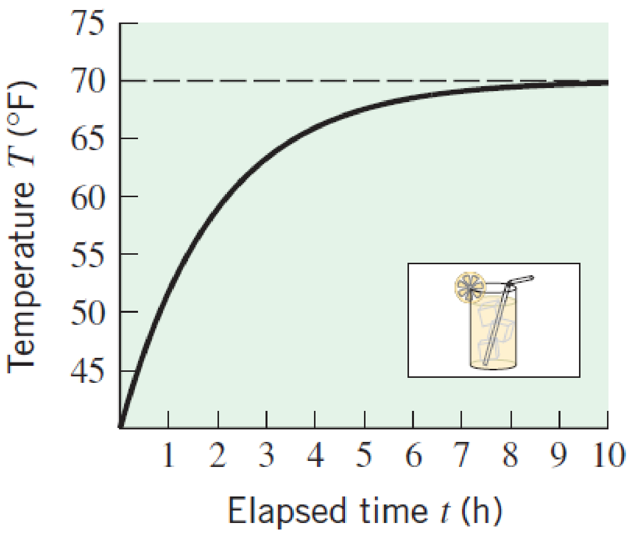


Figure 1-12

Solution. From Definition 4.8.1 the average value of T over the time interval [0, 5] is

Definition (4.8.1) If f is continuous on $[a, b]$, then the ***average value*** (or ***mean value***) of f on $[a, b]$ is defined to be

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$T_{ave} = \frac{1}{5}\int_0^5 (70 - 30e^{-0.5t})\,dt \qquad (1)$$

To evaluate this integral, we make the substitution

$$u = -0.5t \qquad \text{so that} \qquad du = -0.5dt \qquad (\text{or } dt = -2\,du)$$

With this substitution, if

$$t = 0, \qquad u = 0$$

$$t = 5, \qquad u = (-0.5)5 = -2.5$$

Thus, (1) can be expressed as

$$\frac{1}{5}\int_0^{-2.5}(70-30e^u)(-2)\,du = -\frac{2}{5}\int_0^{-2.5}(70-30e^u)\,du$$
$$= -\frac{2}{5}\left[70u-30e^u\right]_{u=0}^{-2.5}$$
$$= -\frac{2}{5}[(-175-30e^{-2.5})-(-30)]$$
$$= 58+12e^{-2.5} \approx 59°\text{F}$$

1.5 L'HÔPITAL'S RULE; INDETERMINATE FORMS

1.5.1 Indeterminate Forms of Type 0/0

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

in which $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$ is called an ***indeterminate form of type* 0/0**. Some examples encountered earlier in the text are

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=2,\qquad \lim_{x\to 0}\frac{\sin x}{x}=1,\qquad \lim_{x\to 0}\frac{1-\cos x}{x}=0$$

L'Hôpital's rule, converts the given indeterminate form into a limit involving derivatives that is often easier to evaluate.

Theorem 1.5 (***L'Hôpital's Rule for Form* 0/0**)

Suppose that f and g are differentiable functions on an open interval containing x = a, except possibly at x = a, and that

$$\lim_{x\to a} f(x)=0 \quad \text{and} \quad \lim_{x\to a} g(x)=0$$

If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ *exists, or if this limit is +∞or −∞, then*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x\to a^-$, $x\to a^+$, $x\to -\infty$, *or as* $x\to +\infty$.

Applying L'Hôpital's Rule

Step 1. Check that the limit of *f(x)/g(x)* is an indeterminate form of type 0/0.

Step 2. Differentiate *f* and *g* separately.

Step 3. Find the limit of *f(x)/g(x)*. If this limit is finite, +∞, or −∞, then it is equal to the limit of *f(x)/g(x)*.

Example 1.24 Find the limit

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

using L'Hôpital's rule, and check the result by factoring.

Solution. The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{\frac{d}{dx}[x^2-4]}{\frac{d}{dx}[x-2]}=\lim_{x\to 2}\frac{2x}{1}=4$$

This agrees with the computation

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=4$$

Example 1.25 In each part confirm that the limit is an indeterminate form of type 0/0, and evaluate it using L'Hôpital's rule.

$$\text{(a)}\ \lim_{x\to 0}\frac{\sin 2x}{x}\qquad \text{(b)}\ \lim_{x\to \pi/2}\frac{1-\sin x}{\cos x}\qquad \text{(c)}\ \lim_{x\to 0}\frac{e^x-1}{x^3}$$

$$\text{(d)}\ \lim_{x\to 0^-}\frac{\tan x}{x^2}\qquad \text{(e)}\ \lim_{x\to 0}\frac{1-\cos x}{x^2}\qquad \text{(f)}\ \lim_{x\to +\infty}\frac{x^{-4/3}}{\sin(1/x)}$$

Solution (a). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{\sin 2x}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}[\sin 2x]}{\frac{d}{dx}[x]}=\lim_{x\to 0}\frac{2\cos 2x}{1}=2$$

Solution (b). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to \pi/2}\frac{1-\sin x}{\cos x}=\lim_{x\to \pi/2}\frac{\frac{d}{dx}[1-\sin x]}{\frac{d}{dx}[\cos x]}=\lim_{x\to \pi/2}\frac{-\cos x}{-\sin x}=\frac{0}{-1}=0$$

Solution (c). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{e^x-1}{x^3}=\lim_{x\to 0}\frac{\frac{d}{dx}[e^x-1]}{\frac{d}{dx}[x^3]}=\lim_{x\to 0}\frac{e^x}{3x^2}=+\infty$$

Solution (d). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0^-}\frac{\tan x}{x^2}=\lim_{x\to 0^-}\frac{\sec^2 x}{2x}=-\infty$$

Solution (e). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}$$

Since the new limit is another indeterminate form of type 0/0, we apply L'Hôpital's rule again:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$$

Solution (f). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to +\infty}\frac{x^{-4/3}}{\sin(1/x)}=\lim_{x\to +\infty}\frac{-\frac{4}{3}x^{-7/3}}{(-1/x^2)\cos(1/x)}=\lim_{x\to +\infty}\frac{\frac{4}{3}x^{-1/3}}{\cos(1/x)}=\frac{0}{1}=0$$

1.5.2 Indeterminate Forms of Type ∞/∞

Theorem 1.6 (***L'Hôpital's Rule for Form*** **∞/∞**)

Suppose that f and g are differentiable functions on an open interval containing x = a, except possibly at x = a, and that

$$\lim_{x\to a} f(x) = \infty \quad \textit{and} \quad \lim_{x\to a} g(x) = \infty$$

If $[\lim_{x\to a} \frac{f'(x)}{g'(x)}]$ *exists, or if this limit is +∞or −∞, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x\to a^-$, $x\to a^+$, $x\to -\infty$, *or as* $x\to +\infty$.

Example 1.26 In each part confirm that the limit is an indeterminate form of type ∞/∞ and apply L'Hôpital's rule.

$$\text{(a)} \quad \lim_{x\to+\infty} \frac{x}{e^x} \qquad \text{(b)} \quad \lim_{x\to 0^+} \frac{\ln x}{\csc x}$$

Solution (a). The numerator and denominator both have a limit of +∞, so we have an indeterminate form of type ∞/∞. Applying L'Hôpital's rule yields

$$\lim_{x\to+\infty} \frac{x}{e^x} = \lim_{x\to+\infty} \frac{1}{e^x} = 0$$

Solution (b). The numerator has a limit of −∞ and the denominator has a limit of +∞, so we have an indeterminate form of type ∞/∞. Applying L'Hôpital's rule yields

$$\lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x}$$

This last limit is again an indeterminate form of type ∞/∞. Moreover, any additional applications of L'Hôpital's rule will yield powers of $1/x$ in the numerator and expressions involving $\csc x$ and $\cot x$ in the denominator; thus, repeated application of L'Hôpital's rule simply produces new indeterminate forms. We must try something else.

$$\lim_{x\to 0^+} \left(-\frac{\sin x}{x} \tan x\right) = -\lim_{x\to 0^+} \frac{\sin x}{x} \cdot \lim_{x\to 0^+} \tan x = -(1)(0) = 0$$

$$\lim_{x\to 0^+} \frac{\ln x}{\csc x} = 0$$

1.5.3 Indeterminate Forms of Type 0.∞

Example 1.27 Evaluate

$$\text{(a)} \quad \lim_{x \to 0^+} x \ln x \qquad \text{(b)} \quad \lim_{x \to \pi/4} (1 - \tan x) \sec 2x$$

Solution (a). The factor x has a limit of 0 and the factor $\ln x$ has a limit of $-\infty$, so the stated problem is an indeterminate form of type $\mathbf{0}\cdot\infty$. There are two possible approaches: we can rewrite the limit as

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} \quad \text{or} \quad \lim_{x \to 0^+} \frac{x}{1/\ln x}$$

the first being an indeterminate form of type ∞/∞ and the second an indeterminate form of type $0/0$. However, the first form is the preferred initial choice because the derivative of $1/x$ is less complicated than the derivative of $1/\ln x$. That choice yields

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$$

Solution (b). The stated problem is an indeterminate form of type $0\cdot\infty$. We will convert it to an indeterminate form of type $0/0$:

$$\lim_{x \to \pi/4} (1 - \tan x) \sec 2x = \lim_{x \to \pi/4} \frac{1 - \tan x}{1/\sec 2x} = \lim_{x \to \pi/4} \frac{1 - \tan x}{\cos 2x}$$

$$= \lim_{x \to \pi/4} \frac{-\sec^2 x}{-2\sin 2x} = \frac{-2}{-2} = 1$$

1.5.4 Indeterminate Forms of Type ∞−∞

Example 1.28 Evaluate

$$\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right).$$

Solution. Both terms have a limit of $+\infty$, so the stated problem is an indeterminate form of type $\infty - \infty$. Combining the two terms yields

$$\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right) = \lim_{x \to 0^+} \frac{\sin x - x}{x \sin x}$$

which is an indeterminate form of type $0/0$. Applying L'Hôpital's rule twice yields

$$\lim_{x\to 0^+} \frac{\sin x - x}{x \sin x} = \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x}$$
$$= \lim_{x\to 0^+} \frac{-\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0$$

1.5.5 Indeterminate Forms of Type 0^0, ∞^0, 1^∞

Example 1.29 Find

$$\lim_{x\to 0}(1+\sin x)^{1/x}$$

Solution. As discussed above, we begin by introducing a dependent variable

$$y = (1 + \sin x)^{1/x}$$

and taking the natural logarithm of both sides:

$$\ln y = \ln(1+\sin x)^{1/x} = \frac{1}{x}\ln(1+\sin x) = \frac{\ln(1+\sin x)}{x}$$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(1+\sin x)}{x}$$

which is an indeterminate form of type 0/0, so by L'Hôpital's rule

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(1+\sin x)}{x} = \lim_{x\to 0} \frac{(\cos x)/(1+\sin x)}{1} = 1$$

Since we have shown that $\ln y \to 1$ as $x \to 0$, the continuity of the exponential function implies that $e^{\ln y} \to e^1$ as $x \to 0$, and this implies that $y \to e$ as $x \to 0$. Thus,

$$\lim_{x\to 0}(1+\sin x)^{1/x} = e$$

1.6 DERIVATIVES AND INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

1.6.1 Inverse Trigonometric Functions

Definitions

1. The ***inverse sine function***, denoted by $\sin^{-1}$, is defined to be the inverse of the restricted sine function

$$\sin x,\ -\pi/2 \leq x \leq \pi/2$$

2. The ***inverse cosine function***, denoted by cos−1, is defined to be the inverse of the restricted cosine function

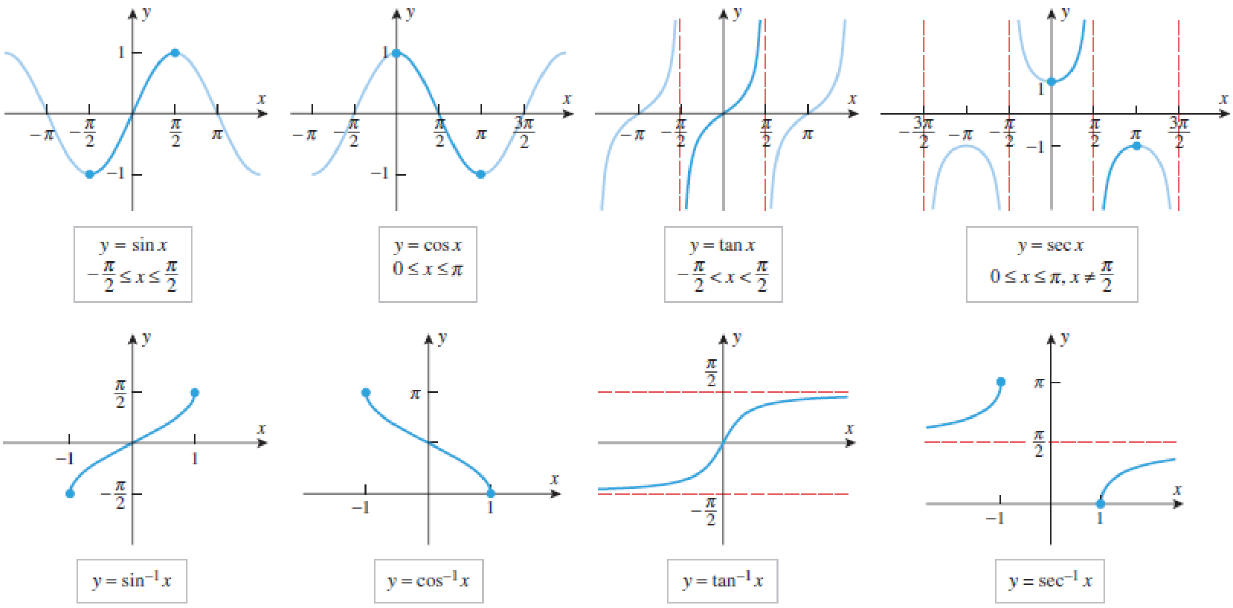
$$\cos x,\ 0 \leq x \leq \pi$$

3. The ***inverse tangent function***, denoted by tan−1, is defined to be the inverse of the restricted tangent function

$$\tan x,\ -\pi/2 < x < \pi/2$$

4. The ***inverse secant function***, denoted by sec−1, is defined to be the inverse of the restricted secant function

$$\sec x,\ 0 \leq x \leq \pi \text{ with } x \neq \pi/2$$



PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

FUNCTION	DOMAIN	RANGE	BASIC RELATIONSHIPS
$\sin^{-1}$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$\sin^{-1}(\sin x) = x$ if $-\pi/2 \le x \le \pi/2$ $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$	$\cos^{-1}(\cos x) = x$ if $0 \le x \le \pi$ $\cos(\cos^{-1} x) = x$ if $-1 \le x \le 1$
$\tan^{-1}$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$	$\tan^{-1}(\tan x) = x$ if $-\pi/2 < x < \pi/2$ $\tan(\tan^{-1} x) = x$ if $-\infty < x < +\infty$
$\sec^{-1}$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$	$\sec^{-1}(\sec x) = x$ if $0 \le x \le \pi, x \ne \pi/2$ $\sec(\sec^{-1} x) = x$ if $\lvert x\rvert \ge 1$

1.6.2 Evaluating Inverse Trigonometric Functions

Example 1 Find exact values of

(a) $\sin^{-1}(1/\sqrt{2})$ (b) $\sin^{-1}(-1)$

by inspection, and confirm your results numerically using a calculating utility.

Solution (a). Because $\sin^{-1}(1/\sqrt{2}) > 0$, we can view $x = \sin^{-1}(1/\sqrt{2})$ as that angle in the first quadrant such that $\sin\theta = 1/\sqrt{2}$. Thus, $\sin^{-1}(1/\sqrt{2}) = \pi/4$. You can confirm this with your calculating utility by showing that $\sin^{-1}(1/\sqrt{2}) \approx 0.785 \approx \pi/4$.

Solution (b). Because $\sin{-1}(-1) < 0$, we can view $x = \sin{-1}(-1)$ as an angle in the fourth quadrant (or an adjacent axis) such that $\sin x = -1$. Thus, $\sin{-1}(-1) = -\pi/2$. You can confirm this with your calculating utility by showing that $\sin{-1}(-1) \approx -1.57 \approx -\pi/2$.

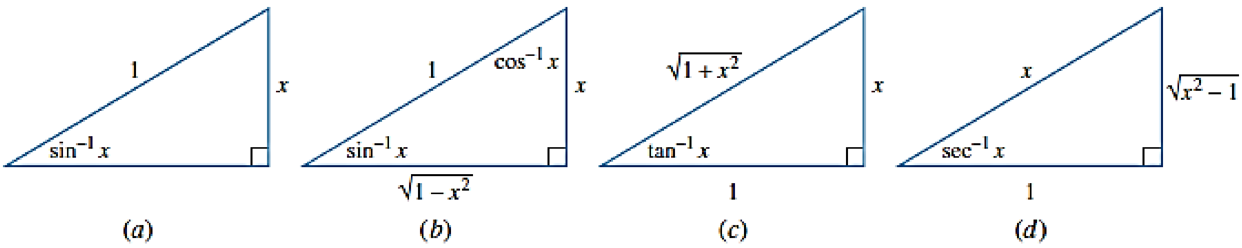
1.6.3 Identities for Inverse Trigonometric Functions

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$
$$\cos(\sin^{-1} x) = \sqrt{1-x^2}$$
$$\sin(\cos^{-1} x) = \sqrt{1-x^2}$$
$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}$$

$$\sec(\tan^{-1} x) = \sqrt{1+x^2}$$
$$\sin(\sec^{-1} x) = \frac{\sqrt{x^2-1}}{x} \quad (x \ge 1)$$

$$\sin(\sec^{-1} x) = \frac{\sqrt{x^2-1}}{|x|} \quad (|x| \ge 1)$$

$$\sin^{-1}(-x) = -\sin^{-1}(x) \quad \text{and} \quad \tan^{-1}(-x) = -\tan^{-1}(x)$$



Example 2 Figure in the below shows a computer-generated graph of $y = \sin^{-1}(\sin x)$. One might think that this graph should be the line $y = x$, since $\sin^{-1}(\sin x) = x$. Why isn't it?

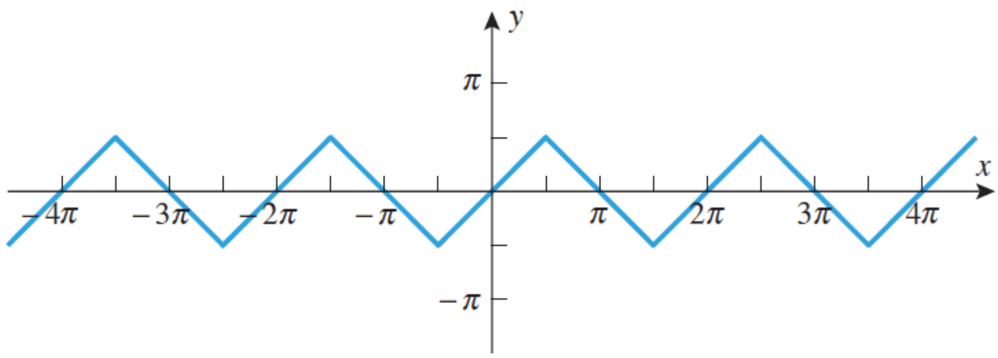
Solution. The relationship $\sin^{-1}(\sin x) = x$ is valid on the interval $-\pi/2 \le x \le \pi/2$, so we can say with certainty that the graphs of $y = \sin^{-1}(\sin x)$ and $y = x$ coincide on this interval (which is confirmed by below Figure). However, outside of this interval the relationship $\sin^{-1}(\sin x) = x$ does not hold. For example, if the quantity x lies in the interval $\pi/2 \le x \le 3\pi/2$, then the quantity $x - \pi$ lies in the interval $-\pi/2 \le x \le \pi/2$, so

$$\sin^{-1}[\sin(x-\pi)] = x - \pi$$

Thus, by using the identity $\sin(x - \pi) = -\sin x$ and the fact that sin−1 is an odd function, we can express sin−1$(\sin x)$ as

$$\sin^{-1}(\sin x) = \sin^{-1}[-\sin(x-\pi)] = -\sin^{-1}[\sin(x-\pi)] = -(x-\pi)$$

This shows that on the interval $\pi/2 \le x \le 3\pi/2$ the graph of $y = \sin^{-1}(\sin x)$ coincides with the line $y = -(x - \pi)$, which has slope -1 and an x-intercept at $x = \pi$. This agrees with below figure.



1.6.4 Derivatives of the Inverse Trigonometric Functions

$$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \frac{d}{dx}[\cos^{-1} u] = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}[\tan^{-1} u] = \frac{1}{1+u^2}\frac{du}{dx} \qquad \frac{d}{dx}[\cot^{-1} u] = -\frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}[\sec^{-1} u] = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad \frac{d}{dx}[\csc^{-1} u] = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

Example 3 Find *dy/dx* if

(a) $y = \sin^{-1}(x^3)$ (b) $y = \sec^{-1}(e^x)$

Solution (a).

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x^3)^2}}(3x^2) = \frac{3x^2}{\sqrt{1-x^6}}$$

Solution (b).

$$\frac{dy}{dx} = \frac{1}{e^x\sqrt{(e^x)^2-1}}(e^x) = \frac{1}{\sqrt{e^{2x}-1}}$$

1.6.5 Integration Formulas

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$$
$$\int \frac{du}{1+u^2} = \tan^{-1} u + C$$
$$\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} |u| + C$$

Example 4 Evaluate

$$\int \frac{dx}{1+3x^2}.$$

Solution. Substituting

$$u = \sqrt{3}x, \quad du = \sqrt{3}\,dx$$

yields

$$\int \frac{dx}{1+3x^2} = \frac{1}{\sqrt{3}}\int \frac{du}{1+u^2} = \frac{1}{\sqrt{3}}\tan^{-1} u + C = \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}x) + C$$

Example 5 Evaluate

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx.$$

Solution. Substituting

$$u = e^x,\ du = e^x\,dx$$

yields

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(e^x) + C$$

Example 6 Evaluate

$$\int \frac{dx}{a^2+x^2}\,dx,$$

where $a \neq 0$ is a constant.

Solution. Some simple algebra and an appropriate u-substitution will allow us to use

$$\int \frac{dx}{a^2+x^2} = \int \frac{a(dx/a)}{a^2(1+(x/a)^2)} = \frac{1}{a}\int \frac{dx/a}{1+(x/a)^2} \qquad \boxed{\begin{array}{c} u = x/a \\ du = dx/a \end{array}}$$

$$= \frac{1}{a}\int \frac{du}{1+u^2} = \frac{1}{a}\tan^{-1} u + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

The method of Example 6 leads to the following generalizations for $a > 0$:

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

Example 7 Evaluate

$$\int \frac{dx}{\sqrt{2-x^2}}.$$

Solution. Applying (previous eq.) with $u = x$ and $a = \sqrt{2}$ yields

$$\int \frac{dx}{\sqrt{2-x^2}} = \sin^{-1}\frac{x}{\sqrt{2}} + C$$

1.7 HYPERBOLIC FUNCTIONS AND HANGING CABLES

1.7.1 Definitions of Hyperbolic Functions

Definitions

Hyperbolic sine	$\sinh x = \dfrac{e^x - e^{-x}}{2}$
Hyperbolic cosine	$\cosh x = \dfrac{e^x + e^{-x}}{2}$
Hyperbolic tangent	$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
Hyperbolic cotangent	$\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
Hyperbolic secant	$\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$
Hyperbolic cosecant	$\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

Example 1

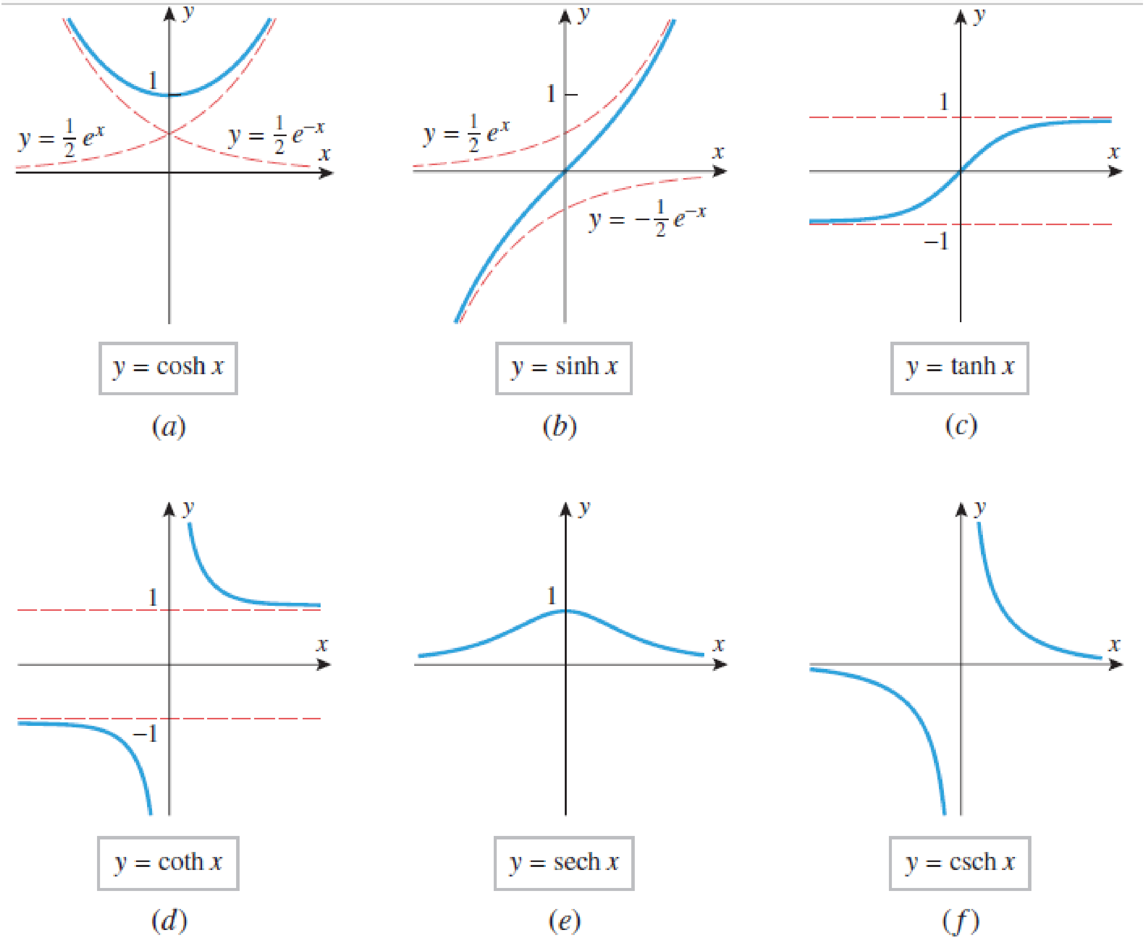
$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0$$
$$\cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1$$
$$\sinh 2 = \frac{e^2 - e^{-2}}{2} \approx 3.6269$$

1.7.2 Graphs of the Hyperbolic Functions

*The graphs of the hyperbolic functions, which are shown in the below figure, can be generated with a graphing utility, but it is worthwhile to observe that the general shape of the graph of $y = \cosh x$ can be obtained by sketching the graphs of $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ separately and adding the corresponding y-coordinates [see part (*a*) of the figure].

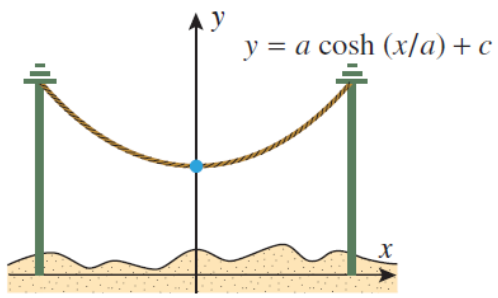
*Similarly, the general shape of the graph of $y = \sinh x$ can be obtained by sketching the graphs of $y = \frac{1}{2}e^x$ and $y = -\frac{1}{2}e^{-x}$ separately and adding corresponding y-coordinates [see part (*b*) of the figure].

*Observe also that $y = \frac{1}{2}e^{x}$ and $y = \frac{1}{2}e^{-x}$ are ***curvilinear asymptotes*** for y = cosh x in the sense that the graph of y = cosh x gets closer and closer to the graph of $y = \frac{1}{2}e^{x}$as $x \to +\infty$ and gets closer and closer to the graph of $y = \frac{1}{2}e^{-x}$ as $x \to -\infty$.



1.7.3 Hanging Cables and Other Applications

Hyperbolic functions arise in vibratory motions inside elastic solids and more generally in many problems where mechanical energy is gradually absorbed by a surrounding medium. They also occur when a homogeneous, flexible cable is suspended between two points, as with a telephone line hanging between two poles. Such a cable forms a curve, called a ***catenary*** (from the Latin *catena*, meaning "chain").



1.7.4 Hyperbolic Identities

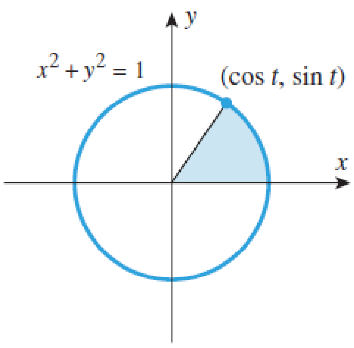
Theorem

$\cosh x + \sinh x = e^x$	$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
$\cosh x - \sinh x = e^{-x}$	$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$
$\cosh^2 x - \sinh^2 x = 1$	$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$
$1 - \tanh^2 x = \operatorname{sech}^2 x$	$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$
$\coth^2 x - 1 = \operatorname{csch}^2 x$	$\sinh 2x = 2 \sinh x \cosh x$
$\cosh(-x) = \cosh x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$
$\sinh(-x) = -\sinh x$	$\cosh 2x = 2\sinh^2 x + 1 = 2\cosh^2 x - 1$

$$x = \cos t, \quad y = \sin t \qquad (0 \le t \le 2\pi)$$

represent the unit circle $x^2 + y^2 = 1$

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$



(a)

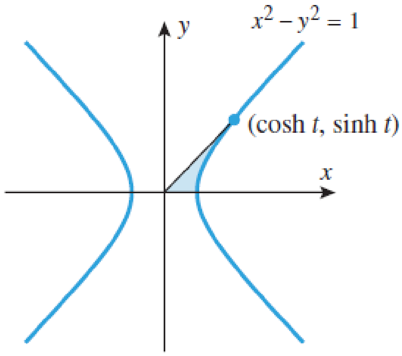
If $0 \le t \le 2\pi$, then the parameter t can be interpreted as the angle in radians from the positive x-axis to the point (cos t, sin t) or, alternatively, as twice the shaded area of the sector in Figure a. Analogously, the parametric equations

$$x = \cosh t,\ y = \sinh t\ (-\infty < t < +\infty)$$

represent a portion of the curve $x^2 - y^2 = 1$, as may be seen by writing

$$x^2 - y^2 = \cosh^2 t - \sinh^2 t = 1$$

and observing that $x = \cosh t > 0$. This curve, which is shown in Figure b, is the right half of a larger curve called the ***unit hyperbola***; this is the reason why the functions in this section are called *hyperbolic* functions.



(b)

1.7.5 Derivative and Integral Formulas

Theorem

$$\frac{d}{dx}[\sinh u] = \cosh u \frac{du}{dx} \qquad \int \cosh u\, du = \sinh u + C$$
$$\frac{d}{dx}[\cosh u] = \sinh u \frac{du}{dx} \qquad \int \sinh u\, du = \cosh u + C$$
$$\frac{d}{dx}[\tanh u] = \operatorname{sech}^2 u \frac{du}{dx} \qquad \int \operatorname{sech}^2 u\, du = \tanh u + C$$
$$\frac{d}{dx}[\coth u] = -\operatorname{csch}^2 u \frac{du}{dx} \qquad \int \operatorname{csch}^2 u\, du = -\coth u + C$$
$$\frac{d}{dx}[\operatorname{sech} u] = -\operatorname{sech} u \tanh u \frac{du}{dx} \qquad \int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$
$$\frac{d}{dx}[\operatorname{csch} u] = -\operatorname{csch} u \coth u \frac{du}{dx} \qquad \int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

Example 2

$$\frac{d}{dx}[\cosh(x^3)] = \sinh(x^3) \cdot \frac{d}{dx}[x^3] = 3x^2 \sinh(x^3)$$
$$\frac{d}{dx}[\ln(\tanh x)] = \frac{1}{\tanh x} \cdot \frac{d}{dx}[\tanh x] = \frac{\operatorname{sech}^2 x}{\tanh x}$$

Example 3

$$\int \sinh^5 x \cosh x\, dx = \tfrac{1}{6}\sinh^6 x + C \qquad \boxed{\begin{array}{c} u = \sinh x \\ du = \cosh x\, dx \end{array}}$$
$$\begin{aligned} \int \tanh x\, dx &= \int \frac{\sinh x}{\cosh x}\, dx \\ &= \ln|\cosh x| + C \qquad \boxed{\begin{array}{c} u = \cosh x \\ du = \sinh x\, dx \end{array}} \\ &= \ln(\cosh x) + C \end{aligned}$$

We were justified in dropping the absolute value signs since $\cosh x > 0$ for all x.

Example 4 A 100 ft wire is attached at its ends to the tops of two 50 ft poles that are positioned 90 ft apart. How high above the ground is the middle of the wire?

Solution. From above, the wire forms a catenary curve with equation

$$y = a\cosh\left(\frac{x}{a}\right) + c$$

where the origin is on the ground midway between the poles. Using Formula for the length of the catenary, we have

$$\begin{aligned}
100 &= \int_{-45}^{45} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= 2\int_{0}^{45} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx && \boxed{\text{By symmetry about the } y\text{-axis}} \\
&= 2\int_{0}^{45} \sqrt{1+\sinh^2\left(\frac{x}{a}\right)}\,dx \\
&= 2\int_{0}^{45} \cosh\left(\frac{x}{a}\right)\,dx && \boxed{\text{By (1) and the fact that } \cosh x > 0} \\
&= 2a\sinh\left(\frac{x}{a}\right)\Big]_0^{45} = 2a\sinh\left(\frac{45}{a}\right)
\end{aligned}$$

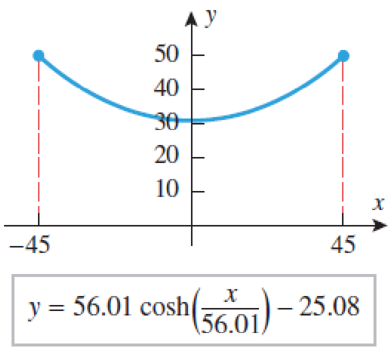
Using a calculating utility's numeric solver to solve

$$100 = 2a\sinh\left(\frac{45}{a}\right)$$

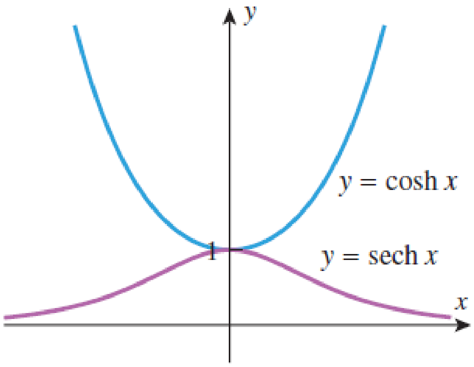
for a gives $a \approx 56.01$. Then

$$50 = y(45) = 56.01\cosh\left(\frac{45}{56.01}\right) + c \approx 75.08 + c$$

so $c \approx -25.08$. Thus, the middle of the wire is $y(0) \approx 56.01 - 25.08 = 30.93$ ft above the ground (below figure).



1.7.6 Inverses of Hyperbolic Functions



With the restriction that $x \geq 0$, the curves $y = \cosh x$ and $y = \operatorname{sech} x$ pass the horizontal line test.

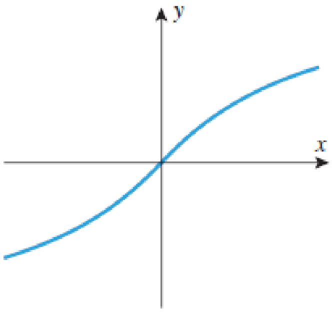
1.7.7 Logarithmic Forms of Inverse Hyperbolic Functions

Theorem *The following relationships hold for all x in the domains of the stated inverse hyperbolic functions*:

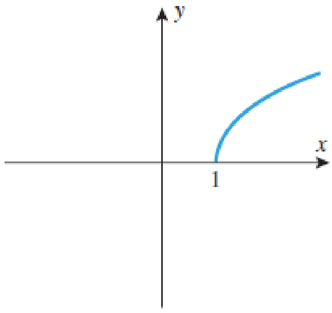
$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \qquad \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$$

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right) \qquad \coth^{-1} x = \frac{1}{2} \ln\left(\frac{x + 1}{x - 1}\right)$$

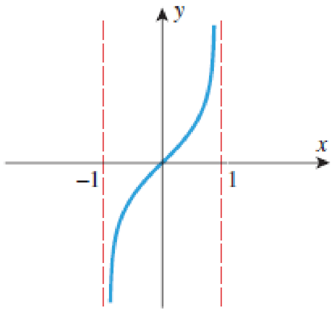
$$\operatorname{sech}^{-1} x = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) \qquad \operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1 + x^2}}{|x|}\right)$$



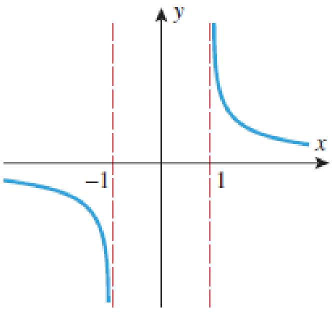
$y = \sinh^{-1} x$



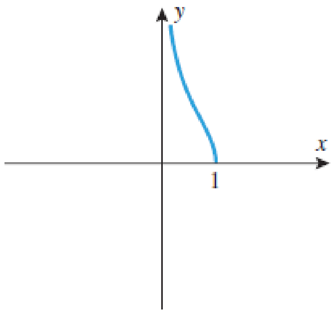
$y = \cosh^{-1} x$



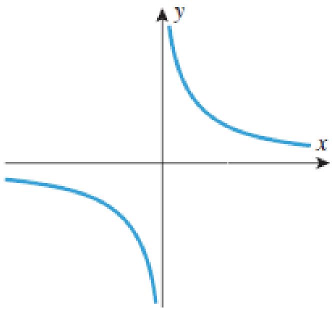
$y = \tanh^{-1} x$



$y = \coth^{-1} x$



$y = \operatorname{sech}^{-1} x$



$y = \operatorname{csch}^{-1} x$

PROPERTIES OF INVERSE HYPERBOLIC FUNCTIONS

FUNCTION	DOMAIN	RANGE	BASIC RELATIONSHIPS
$\sinh^{-1} x$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$\sinh^{-1}(\sinh x) = x$ if $-\infty < x < +\infty$ $\sinh(\sinh^{-1} x) = x$ if $-\infty < x < +\infty$
$\cosh^{-1} x$	$[1, +\infty)$	$[0, +\infty)$	$\cosh^{-1}(\cosh x) = x$ if $x \geq 0$ $\cosh(\cosh^{-1} x) = x$ if $x \geq 1$
$\tanh^{-1} x$	$(-1, 1)$	$(-\infty, +\infty)$	$\tanh^{-1}(\tanh x) = x$ if $-\infty < x < +\infty$ $\tanh(\tanh^{-1} x) = x$ if $-1 < x < 1$
$\coth^{-1} x$	$(-\infty, -1) \cup (1, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$	$\coth^{-1}(\coth x) = x$ if $x < 0$ or $x > 0$ $\coth(\coth^{-1} x) = x$ if $x < -1$ or $x > 1$
$\operatorname{sech}^{-1} x$	$(0, 1]$	$[0, +\infty)$	$\operatorname{sech}^{-1}(\operatorname{sech} x) = x$ if $x \geq 0$ $\operatorname{sech}(\operatorname{sech}^{-1} x) = x$ if $0 < x \leq 1$
$\operatorname{csch}^{-1} x$	$(-\infty, 0) \cup (0, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$	$\operatorname{csch}^{-1}(\operatorname{csch} x) = x$ if $x < 0$ or $x > 0$ $\operatorname{csch}(\operatorname{csch}^{-1} x) = x$ if $x < 0$ or $x > 0$

1.7.8 Derivatives and Integrals Involving Inverse Hyperbolic Functions

Theorem

$$\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx} \qquad \frac{d}{dx}(\coth^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| > 1$$

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad u > 1 \qquad \frac{d}{dx}(\operatorname{sech}^{-1} u) = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad 0 < u < 1$$

$$\frac{d}{dx}(\tanh^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1 \qquad \frac{d}{dx}(\operatorname{csch}^{-1} u) = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \quad u \neq 0$$

Example 4:

$$\frac{d}{dx}[\sinh^{-1} x] = \frac{d}{dx}[\ln(x+\sqrt{x^2+1})] = \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)$$
$$= \frac{\sqrt{x^2+1}+x}{(x+\sqrt{x^2+1})(\sqrt{x^2+1})} = \frac{1}{\sqrt{x^2+1}}$$

Theorem

$$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \;\; or \;\; \ln(u+\sqrt{u^2+a^2}) + C$$

$$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \;\; or \;\; \ln(u+\sqrt{u^2-a^2}) + C, \quad u > a$$

$$\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C, & |u| < a \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & |u| > a \end{cases} \;\; or \;\; \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C, \quad |u| \neq a$$

$$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left|\frac{u}{a}\right| + C \;\; or \;\; -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-u^2}}{|u|}\right) + C, \quad 0 < |u| < a$$

$$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C \;\; or \;\; -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2+u^2}}{|u|}\right) + C, \quad u \neq 0$$

Example 5: Evaluate

$$\int \frac{dx}{\sqrt{4x^2-9}}, x > \frac{3}{2}.$$

Solution. Let $u = 2x$. Thus, $du = 2\ dx$ and

$$\int \frac{dx}{\sqrt{4x^2-9}} = \frac{1}{2}\int \frac{2\,dx}{\sqrt{4x^2-9}} = \frac{1}{2}\int \frac{du}{\sqrt{u^2-3^2}}$$
$$= \frac{1}{2}\cosh^{-1}\left(\frac{u}{3}\right) + C = \frac{1}{2}\cosh^{-1}\left(\frac{2x}{3}\right) + C$$

Alternatively, we can use the logarithmic equivalent of $\cosh^{-1}(2x/3)$,

$$\cosh^{-1}\left(\frac{2x}{3}\right) = \ln(2x + \sqrt{4x^2-9}) - \ln 3$$

(verify), and express the answer as

$$\int \frac{dx}{\sqrt{4x^2-9}} = \frac{1}{2}\ln(2x + \sqrt{4x^2-9}) + C$$

University of Anbar

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LECTURE NOTE

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CALCULUS II

Chapter Two

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CHAPTER TWO

PRINCIPLES OF INTEGRAL EVALUATION

2.1 A REVIEW OF FAMILIAR INTEGRATION FORMULAS

CONSTANTS, POWERS, EXPONENTIALS

1. $\int du = u + C$

2. $\int a\, du = a \int du = au + C$

3. $\int u^r\, du = \frac{u^{r+1}}{r+1} + C,\ r \neq -1$

4. $\int \frac{du}{u} = \ln|u| + C$

5. $\int e^u\, du = e^u + C$

6. $\int b^u\, du = \frac{b^u}{\ln b} + C,\ b > 0, b \neq 1$

TRIGONOMETRIC FUNCTIONS

7. $\int \sin u\, du = -\cos u + C$

8. $\int \cos u\, du = \sin u + C$

9. $\int \sec^2 u\, du = \tan u + C$

10. $\int \csc^2 u\, du = -\cot u + C$

11. $\int \sec u \tan u\, du = \sec u + C$

12. $\int \csc u \cot u\, du = -\csc u + C$

13. $\int \tan u\, du = -\ln|\cos u| + C$

14. $\int \cot u\, du = \ln|\sin u| + C$

HYPERBOLIC FUNCTIONS

15. $\int \sinh u\, du = \cosh u + C$

16. $\int \cosh u\, du = \sinh u + C$

17. $\int \operatorname{sech}^2 u\, du = \tanh u + C$

18. $\int \operatorname{csch}^2 u\, du = -\coth u + C$

19. $\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$

20. $\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$

ALGEBRAIC FUNCTIONS *(a > 0)*

21. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C \qquad (|u| < a)$

22. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

23. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \qquad (0 < a < |u|)$

24. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \ln(u+\sqrt{u^2+a^2})+C$

25. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right|+C \qquad (0<a<|u|)$

26. $\displaystyle\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right|+C$

27. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right|+C \qquad (0<|u|<a)$

28. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right|+C$

2.2 INTEGRATION BY PARTS

2.2.1 The Product Rule and Integration by Parts

$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx \qquad (1)$$

This formula allows us to integrate $f(x)g(x)$ by integrating $f(x)G(x)$ instead, and in many cases the net effect is to replace a difficult integration with an easier one. The application of this formula is called ***integration by parts***.

In practice, we usually rewrite (1) by letting

$$u = f(x), \qquad du = f'(x)\,dx$$

$$v = G(x), \qquad dv = G'(x)\,dx = g(x)\,dx$$

This yields the following alternative form for (1):

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

Example 2.1 Use integration by parts to evaluate $\int x \cos x\, dx$.

Solution. We will apply Formula (2). The first step is to make a choice for u and dv to put the given integral in the form $u\,dv$. We will let

$$u = x \quad \text{and} \quad dv = \cos x\,dx$$

(Other possibilities will be considered later.) The second step is to compute du from u and v from dv. This yields

$$du = dx \quad \text{and} \quad v = \int dv = \int \cos x\,dx = \sin x$$

The third step is to apply Formula (2). This yields

$$\int \underbrace{x}_{u} \underbrace{\cos x\,dx}_{dv} = \underbrace{x}_{u} \underbrace{\sin x}_{v} - \int \underbrace{\sin x}_{v} \underbrace{dx}_{du}$$
$$= x\sin x - (-\cos x) + C = x\sin x + \cos x + C$$

2.2.2 Guidelines for Integration by Parts

For the integral *x* cos*x dx* in Example 2.1, both goals were achieved by letting $u = x$ and $dv = \cos x\,dx$. In contrast, $u = \cos x$ would not have been a good first choice in that example, since $du/dx = -\sin x$ is no simpler than u. Indeed, had we chosen

$$u = \cos x \qquad dv = x\,dx$$
$$du = -\sin x\,dx \qquad v = \int x\,dx = \frac{x^2}{2}$$

then we would have obtained

$$\int x\cos x\,dx = \frac{x^2}{2}\cos x - \int \frac{x^2}{2}(-\sin x)\,dx = \frac{x^2}{2}\cos x + \frac{1}{2}\int x^2 \sin x\,dx$$

There is another useful strategy for choosing u and dv that can be applied when the integrand is a product of two functions from *different* categories in the list

Logarithmic, **I**nverse trigonometric, **A**lgebraic, **T**rigonometric, **E**xponential

In this case you will often be successful if you take ***u*** to be the function whose category occurs earlier in the list and take ***dv*** to be the rest of the integrand. The acronym **LIATE** will help you to remember the order. The method does not work all the time, but it works often enough to be useful.

Example 2.2 Evaluate $\int xe^x\,dx$.

Solution. In this case the integrand is the product of the algebraic function x with the exponential function e^x. According to LIATE we should let

$$u = x \quad \text{and} \quad dv = e^x\,dx$$

$$du = dx \quad \text{and} \quad v = \int e^x\,dx = e^x$$

$$\int xe^x\,dx = \int u\,dv = uv - \int v\,du = xe^x - \int e^x\,dx = xe^x - e^x + C$$

Example 2.3 Evaluate $\int \ln x\,dx$.

Solution. One choice is to let $u = 1$ and $dv = \ln x\,dx$. But with this choice finding v is equivalent to evaluating $\int \ln x\,dx$ and we have gained nothing. Therefore, the only reasonable choice is to let

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x}\,dx \qquad v = \int dx = x$$

$$\int \ln x\,dx = \int u\,dv = uv - \int v\,du = x\ln x - \int dx = x\ln x - x + C$$

2.2.3 Repeated Integration by Parts

Example 2.4 Evaluate $\int x^2\, e^{-x} dx$.

Solution: Let

$$u = x^2, \quad dv = e^{-x}\,dx, \quad du = 2x\,dx, \quad v = \int e^{-x}\,dx = -e^{-x}$$

$$\begin{aligned}\int x^2 e^{-x}\,dx &= \int u\,dv = uv - \int v\,du \\ &= x^2(-e^{-x}) - \int -e^{-x}(2x)\,dx \\ &= -x^2 e^{-x} + 2\int xe^{-x}\,dx\end{aligned}$$

The last integral is similar to the original except that we have replaced x^2 by x. Another integration by parts applied to $\int x\, e^{-x}\, dx$ will complete the problem. We let

$$u = x, \quad dv = e^{-x}\,dx, \quad du = dx, \quad v = \int e^{-x}\,dx = -e^{-x}$$

so that

$$\int xe^{-x}\,dx = x(-e^{-x}) - \int -e^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

Finally, substituting this into the last line yields

$$\begin{aligned}\int x^2 e^{-x}\,dx &= -x^2 e^{-x} + 2\int xe^{-x}\,dx = -x^2 e^{-x} + 2(-xe^{-x} - e^{-x}) + C \\ &= -(x^2 + 2x + 2)e^{-x} + C\end{aligned}$$

The LIATE method suggests that integrals of the form

$$\int e^{ax} \sin bx\,dx \quad \text{and} \quad \int e^{ax} \cos bx\,dx$$

can be evaluated by letting $u = \sin bx$ or $u = \cos bx$ and $dv = e^{ax}\,dx$. However, this will require a technique that deserves special attention.

Example 2.5 Evaluate $\int e^x \cos x \, dx$.

Solution: Let

$$u = \cos x, \quad dv = e^x \, dx, \quad du = -\sin x \, dx, \quad v = \int e^x \, dx = e^x$$

Thus,

$$\int e^x \cos x \, dx = \int u \, dv = uv - \int v \, du = e^x \cos x + \int e^x \sin x \, dx \tag{1}$$

Since the integral $\int e^x \sin x \, dx$ is similar in form to the original integral $\int e^x \cos x \, dx$, it seems that nothing has been accomplished. However, let us integrate this new integral by parts. We let

$$u = \sin x, \quad dv = e^x \, dx, \quad du = \cos x \, dx, \quad v = \int e^x \, dx = e^x$$

Thus,
$$\int e^x \sin x \, dx = \int u \, dv = uv - \int v \, du = e^x \sin x - \int e^x \cos x \, dx$$

Together with Equation (1) this yields

$$\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

which is an equation we can solve for the unknown integral. We obtain

$$2 \int e^x \cos x \, dx = e^x \cos x + e^x \sin x$$

and hence

$$\int e^x \cos x \, dx = \tfrac{1}{2} e^x \cos x + \tfrac{1}{2} e^x \sin x + C$$

2.2.4 A Tabular Method for Repeated Integration by Parts

Integrals of the form

$$\int p(x) f(x) \, dx$$

Tabular Integration by Parts

Step 1. Differentiate *p(x)* repeatedly until you obtain 0, and list the results in the first column.

Step 2. Integrate *f(x)* repeatedly and list the results in the second column.

Step 3. Draw an arrow from each entry in the first column to the entry that is one row down in the second column.

Step 4. Label the arrows with alternating + and − signs, starting with a +.

Step 5. For each arrow, form the product of the expressions at its tip and tail and then multiply that product by +1 or −1 in accordance with the sign on the arrow.

Add the results to obtain the value of the integral.

Example 2.6 Evaluate $\int x^2\sqrt{x-1}\,dx$ using tabular integration by parts.

Solution:

REPEATED DIFFERENTIATION		REPEATED INTEGRATION
x^2	$+$	$(x-1)^{1/2}$
$2x$	$-$	$\frac{2}{3}(x-1)^{3/2}$
2	$+$	$\frac{4}{15}(x-1)^{5/2}$
0		$\frac{8}{105}(x-1)^{7/2}$

$$\int x^2\sqrt{x-1}\,dx = \tfrac{2}{3}x^2(x-1)^{3/2} - \tfrac{8}{15}x(x-1)^{5/2} + \tfrac{16}{105}(x-1)^{7/2} + C$$

2.2.5 Integration by Parts for Definite Integrals

For definite integrals the formula is

$$\int_a^b u\,dv = uv\Big]_a^b - \int_a^b v\,du$$

Example 2.7: Evaluate $\int_0^1 tan^{-1}x\,dx$

Solution: Let

$$u = \tan^{-1}x, \quad dv = dx, \quad du = \frac{1}{1+x^2}\,dx, \quad v = x$$

Thus,

$$\int_0^1 \tan^{-1}x\,dx = \int_0^1 u\,dv = uv\Big]_0^1 - \int_0^1 v\,du$$

The limits of integration refer to x; that is, $x = 0$ and $x = 1$.

$$= x\tan^{-1}x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

But

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx = \frac{1}{2}\ln(1+x^2)\Big]_0^1 = \frac{1}{2}\ln 2$$

so

$$\int_0^1 \tan^{-1}x\,dx = x\tan^{-1}x\Big]_0^1 - \frac{1}{2}\ln 2 = \left(\frac{\pi}{4} - 0\right) - \frac{1}{2}\ln 2 = \frac{\pi}{4} - \ln\sqrt{2}$$

2.3 INTEGRATING TRIGONOMETRIC FUNCTIONS

2.3.1 Integrating Powers of Sine and Cosine

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

In the case where $n = 2$, these formulas yield

$$\int \sin^2 x\,dx = -\tfrac{1}{2}\sin x\cos x + \tfrac{1}{2}\int dx = \tfrac{1}{2}x - \tfrac{1}{2}\sin x\cos x + C$$

$$\int \cos^2 x\,dx = \tfrac{1}{2}\cos x\sin x + \tfrac{1}{2}\int dx = \tfrac{1}{2}x + \tfrac{1}{2}\sin x\cos x + C$$

Alternative forms of these integration formulas can be derived from the trigonometric identities

$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x) \quad \text{and} \quad \cos^2 x = \tfrac{1}{2}(1+\cos 2x)$$

which follow from the double-angle formulas

$$\cos 2x = 1 - 2\sin^2 x \quad \text{and} \quad \cos 2x = 2\cos^2 x - 1$$

These identities yield

$$\int \sin^2 x\,dx = \tfrac{1}{2}\int (1-\cos 2x)\,dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + C$$

$$\int \cos^2 x\,dx = \tfrac{1}{2}\int (1+\cos 2x)\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + C$$

$$\int \sin^3 x\,dx = \tfrac{1}{3}\cos^3 x - \cos x + C$$

$$\int \cos^3 x\,dx = \sin x - \tfrac{1}{3}\sin^3 x + C$$

$$\int \sin^4 x\,dx = \tfrac{3}{8}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C$$

$$\int \cos^4 x\,dx = \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C$$

2.3.2 Integrating Products of Sines and Cosines

If *m* and *n* are positive integers, then the integral

$$\int \sin^m x \ \cos^n x \, dx$$

can be evaluated by one of the three procedures stated in Table 2-1, depending on whether *m* and *n* are odd or even.

Table 2-1
INTEGRATING PRODUCTS OF SINES AND COSINES

$\int \sin^m x \cos^n x \, dx$	PROCEDURE	RELEVANT IDENTITIES
n odd	• Split off a factor of $\cos x$. • Apply the relevant identity. • Make the substitution $u = \sin x$.	$\cos^2 x = 1 - \sin^2 x$
m odd	• Split off a factor of $\sin x$. • Apply the relevant identity. • Make the substitution $u = \cos x$.	$\sin^2 x = 1 - \cos^2 x$
$\begin{cases} m \text{ even} \\ n \text{ even} \end{cases}$	• Use the relevant identities to reduce the powers on $\sin x$ and $\cos x$.	$\begin{cases} \sin^2 x = \frac{1}{2}(1 - \cos 2x) \\ \cos^2 x = \frac{1}{2}(1 + \cos 2x) \end{cases}$

Example 2.8 Evaluate

$$\text{(a)} \int \sin^4 x \cos^5 x \, dx \qquad \text{(b)} \int \sin^4 x \cos^4 x \, dx$$

Solution (a). Since $n = 5$ is odd, we will follow the first procedure in Table 2-1:

$$\begin{aligned}
\int \sin^4 x \cos^5 x \, dx &= \int \sin^4 x \cos^4 x \cos x \, dx \\
&= \int \sin^4 x (1 - \sin^2 x)^2 \cos x \, dx \\
&= \int u^4 (1 - u^2)^2 \, du \\
&= \int (u^4 - 2u^6 + u^8) \, du \\
&= \tfrac{1}{5} u^5 - \tfrac{2}{7} u^7 + \tfrac{1}{9} u^9 + C \\
&= \tfrac{1}{5} \sin^5 x - \tfrac{2}{7} \sin^7 x + \tfrac{1}{9} \sin^9 x + C
\end{aligned}$$

Solution (b). Since $m = n = 4$, both exponents are even, so we will follow the third procedure in Table 2-1:

$$\begin{aligned}
\int \sin^4 x \cos^4 x\,dx &= \int (\sin^2 x)^2 (\cos^2 x)^2\,dx \\
&= \int \left(\tfrac{1}{2}[1-\cos 2x]\right)^2 \left(\tfrac{1}{2}[1+\cos 2x]\right)^2\,dx \\
&= \tfrac{1}{16}\int (1-\cos^2 2x)^2\,dx \\
&= \tfrac{1}{16}\int \sin^4 2x\,dx && \boxed{\text{Note that this can be obtained more directly from the original integral using the identity } \sin x\cos x = \tfrac{1}{2}\sin 2x.} \\
&= \tfrac{1}{32}\int \sin^4 u\,du && \boxed{u = 2x,\ du = 2\,dx \text{ or } dx = \tfrac{1}{2}\,du} \\
&= \tfrac{1}{32}\left(\tfrac{3}{8}u - \tfrac{1}{4}\sin 2u + \tfrac{1}{32}\sin 4u\right) + C && \boxed{\text{Formula (13)}} \\
&= \tfrac{3}{128}x - \tfrac{1}{128}\sin 4x + \tfrac{1}{1024}\sin 8x + C
\end{aligned}$$

Integrals of the form

$$\int \sin mx \cos nx\,dx, \qquad \int \sin mx \sin nx\,dx, \qquad \int \cos mx \cos nx\,dx$$

can be found by using the trigonometric identities

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

Example 2.9 Evaluate $\int \sin 7x \ \ \cos 3x \ \ dx$

Solution:

$$\int \sin 7x\cos 3x\,dx = \tfrac{1}{2}\int (\sin 4x + \sin 10x)\,dx = -\tfrac{1}{8}\cos 4x - \tfrac{1}{20}\cos 10x + C$$

2.3.3 Integrating Powers of Tangent and Secant

$$\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$$

$$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

In the case where n is odd, the exponent can be reduced to 1, leaving us with the problem of integrating $\tan x$ or $\sec x$. These integrals are given by

$$\int \tan x \, dx = \ln|\sec x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \tan^2 x \, dx = \tan x - x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \tan^3 x \, dx = \tfrac{1}{2}\tan^2 x - \ln|\sec x| + C$$

$$\int \sec^3 x \, dx = \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\ln|\sec x + \tan x| + C$$

2.3.4 Integrating Products of Tangents and Secants

If m and n are positive integers, then the integral

$$\int \tan^m x \sec^n x \, dx$$

can be evaluated by one of the three procedures stated in Table 2-2, depending on whether m and n are odd or even.

Table 2-2

INTEGRATING PRODUCTS OF TANGENTS AND SECANTS

$\int \tan^m x \sec^n x \, dx$	PROCEDURE	RELEVANT IDENTITIES
n even	• Split off a factor of $\sec^2 x$. • Apply the relevant identity. • Make the substitution $u = \tan x$.	$\sec^2 x = \tan^2 x + 1$
m odd	• Split off a factor of $\sec x \tan x$. • Apply the relevant identity. • Make the substitution $u = \sec x$.	$\tan^2 x = \sec^2 x - 1$
$\begin{cases} m \text{ even} \\ n \text{ odd} \end{cases}$	• Use the relevant identities to reduce the integrand to powers of $\sec x$ alone. • Then use the reduction formula for powers of $\sec x$.	$\tan^2 x = \sec^2 x - 1$

Example 2.10 Evaluate

(a) $\int \tan^2 x \sec^4 x \, dx$ (b) $\int \tan^3 x \sec^3 x \, dx$ (c) $\int \tan^2 x \sec x \, dx$

Solution (a). Since n = 4 is even, we will follow the first procedure in Table 2-2:

$$\begin{aligned}\int \tan^2 x \sec^4 x \, dx &= \int \tan^2 x \sec^2 x \sec^2 x \, dx \\ &= \int \tan^2 x (\tan^2 x + 1) \sec^2 x \, dx \\ &= \int u^2(u^2+1) \, du \\ &= \tfrac{1}{5}u^5 + \tfrac{1}{3}u^3 + C = \tfrac{1}{5}\tan^5 x + \tfrac{1}{3}\tan^3 x + C\end{aligned}$$

Solution (b). Since m = 3 is odd, we will follow the second procedure in Table 2-2:

$$\begin{aligned}\int \tan^3 x \sec^3 x \, dx &= \int \tan^2 x \sec^2 x (\sec x \tan x) \, dx \\ &= \int (\sec^2 x - 1) \sec^2 x (\sec x \tan x) \, dx \\ &= \int (u^2 - 1) u^2 \, du \\ &= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{5}\sec^5 x - \tfrac{1}{3}\sec^3 x + C\end{aligned}$$

Solution (c). Since m = 2 is even and n = 1 is odd, we will follow the third procedure in Table 2-2:

$$\begin{aligned}\int \tan^2 x \sec x \, dx &= \int (\sec^2 x - 1) \sec x \, dx \\ &= \int \sec^3 x \, dx - \int \sec x \, dx \qquad \boxed{\text{See (26) and (22)}} \\ &= \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\ln|\sec x + \tan x| - \ln|\sec x + \tan x| + C \\ &= \tfrac{1}{2}\sec x \tan x - \tfrac{1}{2}\ln|\sec x + \tan x| + C\end{aligned}$$

2.3.5 An Alternative Method for Integrating Powers of Sine, Cosine, Tangent, and Secant

The methods in Tables 2-1 and 2-2 can sometimes be applied if m = 0 or n = 0 to integrate positive integer powers of sine, cosine, tangent, and secant without reduction formulas. For example, instead of using the reduction formula to integrate $\sin^3 x$, we can apply the second procedure in Table 2-1:

Example 2.11

$$\begin{aligned}\int \sin^3 x\,dx &= \int (\sin^2 x)\sin x\,dx \\ &= \int (1-\cos^2 x)\sin x\,dx \qquad \boxed{\begin{aligned}u &= \cos x\\ du &= -\sin x\,dx\end{aligned}} \\ &= -\int (1-u^2)\,du \\ &= \tfrac{1}{3}u^3 - u + C = \tfrac{1}{3}\cos^3 x - \cos x + C\end{aligned}$$

2.4 TRIGONOMETRIC SUBSTITUTIONS

2.4.1 The Method of Trigonometric Substitution

$$\sqrt{a^2-x^2}, \quad \sqrt{x^2+a^2}, \quad \sqrt{x^2-a^2}$$

a is a positive constant.

The basic idea for evaluating such integrals is to make a substitution for x that will eliminate the radical. For example, to eliminate the radical in the expression $\sqrt{a^2-x^2}$, we can make the substitution

$$x = a\sin\theta, \quad -\pi/2 \le \theta \le \pi/2 \tag{1}$$

which yields

$$\begin{aligned}\sqrt{a^2-x^2} &= \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} \\ &= a\sqrt{\cos^2\theta} = a|\cos\theta| = a\cos\theta \qquad \boxed{\cos\theta \ge 0 \text{ since } -\pi/2 \le \theta \le \pi/2}\end{aligned}$$

Example 2.12: Evaluate

$$\int \frac{dx}{x^2\sqrt{4-x^2}}.$$

Solution: To eliminate the radical we make the substitution

$$x = 2\sin\theta, \quad dx = 2\cos\theta\, d\theta$$

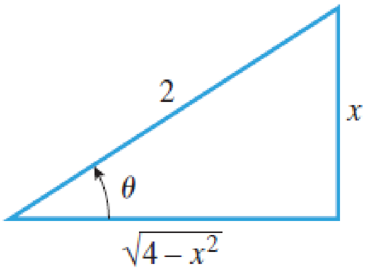
This yields

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2\sqrt{4-4\sin^2\theta}}$$

$$= \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2(2\cos\theta)} = \frac{1}{4}\int \frac{d\theta}{\sin^2\theta}$$

$$= \frac{1}{4}\int \csc^2\theta\, d\theta = -\frac{1}{4}\cot\theta + C$$

At this point we have completed the integration; however, because the original integral was expressed in terms of x, it is desirable to express $\cot\theta$ in terms of x as well. This can be done using trigonometric identities, but the expression can also be obtained by writing substitution u $x = 2\sin\theta$ as $\sin\theta = x/2$ and representing it geometrically as in Figure 2-1.



$\boxed{x = 2\sin\theta}$

Figure 2-1

From that figure we obtain

$$\cot\theta = \frac{\sqrt{4-x^2}}{x}$$

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C$$

TRIGONOMETRIC SUBSTITUTIONS

EXPRESSION IN THE INTEGRAND	SUBSTITUTION	RESTRICTION ON θ	SIMPLIFICATION
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$-\pi/2 \le \theta \le \pi/2$	$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$-\pi/2 < \theta < \pi/2$	$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\begin{cases} 0 \le \theta < \pi/2 & (\text{if } x \ge a) \\ \pi/2 < \theta \le \pi & (\text{if } x \le -a) \end{cases}$	$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

Example 2.13: Evaluate

$$\int \frac{\sqrt{x^2-25}}{x}\,dx, \text{ assuming that } x \geq 5.$$

Solution. The integrand involves a radical of the form $\sqrt{x^2-a^2}$ with $a=5$, so from Table 7.4.1 we make the substitution

$$x = 5\sec\theta, \quad 0 \leq \theta < \pi/2$$

$$\frac{dx}{d\theta} = 5\sec\theta\tan\theta \quad \text{or} \quad dx = 5\sec\theta\tan\theta\,d\theta$$

Thus,

$$\begin{aligned}
\int \frac{\sqrt{x^2-25}}{x}\,dx &= \int \frac{\sqrt{25\sec^2\theta-25}}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta \\
&= \int \frac{5|\tan\theta|}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta \\
&= 5\int \tan^2\theta\,d\theta \qquad \boxed{\tan\theta \geq 0 \text{ since } 0 \leq \theta < \pi/2} \\
&= 5\int (\sec^2\theta - 1)\,d\theta = 5\tan\theta - 5\theta + C
\end{aligned}$$

To express the solution in terms of x, we will represent the substitution $x = 5\sec\theta$ geometrically by the triangle in Figure 2-2, from which we obtain

$$\tan\theta = \frac{\sqrt{x^2-25}}{5}$$

From this and the fact that the substitution can be expressed as $\theta = \sec^{-1}(x/5)$, we obtain

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right) + C$$

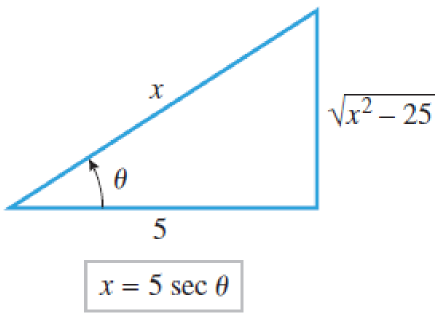


Figure 2-2

2.4.2 Integrals Involving ax^2+bx+c

Integrals that involve a quadratic expression ax^2+bx+c, where $a \neq 0$ and $b \neq 0$, can often be evaluated by first completing the square, then making an appropriate substitution. The following example illustrates this idea.

Example 2.14: Evaluate

$$\int \frac{x}{x^2-4x+8}\,dx.$$

Solution. Completing the square yields

$$x^2-4x+8=(x^2-4x+4)+8-4=(x-2)^2+4$$

Thus, the substitution

$$u=x-2, \quad du=dx$$

yields

$$\begin{aligned}\int \frac{x}{x^2-4x+8}\,dx &= \int \frac{x}{(x-2)^2+4}\,dx = \int \frac{u+2}{u^2+4}\,du \\ &= \int \frac{u}{u^2+4}\,du + 2\int \frac{du}{u^2+4} \\ &= \frac{1}{2}\int \frac{2u}{u^2+4}\,du + 2\int \frac{du}{u^2+4} \\ &= \frac{1}{2}\ln(u^2+4)+2\left(\frac{1}{2}\right)\tan^{-1}\frac{u}{2}+C \\ &= \frac{1}{2}\ln[(x-2)^2+4]+\tan^{-1}\left(\frac{x-2}{2}\right)+C\end{aligned}$$

2.5 INTEGRATING RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

2.5.1 Partial Fractions

$$\frac{2}{x-4}+\frac{3}{x+1}=\frac{2(x+1)+3(x-4)}{(x-4)(x+1)}=\frac{5x-10}{x^2-3x-4}$$

$$\int \frac{5x-10}{x^2-3x-4}\,dx=\int \frac{2}{x-4}\,dx+\int \frac{3}{x+1}\,dx=2\ln|x-4|+3\ln|x+1|+C$$

$$\frac{5x-10}{(x-4)(x+1)}=\frac{A}{x-4}+\frac{B}{x+1}$$

$$5x-10=A(x+1)+B(x-4)$$

$$5x-10=(A+B)x+(A-4B)$$

$$\begin{aligned} A+\ \ B &= \ \ 5 \\ A-4B &= -10 \end{aligned}$$

$$A = 2 \text{ and } B = 3$$

$$\frac{P(x)}{Q(x)}=F_1(x)+F_2(x)+\cdots+F_n(x)$$

where $F_1(x)$, $F_2(x)$, . . . , $F_n(x)$ are rational functions of the form

$$\frac{A}{(ax+b)^k} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^k}$$

2.5.2 Finding the Form of a Partial Fraction Decomposition

The first step in finding the form of the partial fraction decomposition of a proper rational function $P(x)/Q(x)$ is to factor $Q(x)$ completely into ***linear*** and irreducible ***quadratic*** factors, and then collect all repeated factors so that $Q(x)$ is expressed as a product of *distinct* factors of the form

$$(ax+b)^m \text{ and } (ax^2+bx+c)^m$$

A. Linear factors

Linear factor rule

For each factor of the form $(ax + b)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}$$

where $A_1, A_2, \ldots, A_m$ are constants to be determined. In the case where $m = 1$, only the first term in the sum appears.

Example 2.15: Evaluate

$$\int \frac{dx}{x^2+x-2}.$$

Solution. The integrand is a proper rational function that can be written as

$$\frac{1}{x^2+x-2} = \frac{1}{(x-1)(x+2)}$$

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x-1)$$

$$\frac{1}{(x-1)(x+2)} = \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}}{x+2}$$

$$\begin{aligned}\int \frac{dx}{(x-1)(x+2)} &= \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{dx}{x+2} \\ &= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C = \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C\end{aligned}$$

Example 2.16: Evaluate

$$\int \frac{2x+4}{x^3-2x^2}\,dx.$$

Solution. The integrand can be rewritten as

$$\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)}$$

Although x^2 is a quadratic factor, it is *not* irreducible since $x^2 = xx$. Thus, by the linear factor rule, x^2 introduces two terms (since $m = 2$) of the form

$$\frac{A}{x} + \frac{B}{x^2}$$

$$\frac{C}{x-2}$$

$$\frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$2x + 4 = Ax(x-2) + B(x-2) + Cx^2$$

$$2x + 4 = (A + C)x^2 + (-2A + B)x - 2B$$

$$A + C = 0 \qquad \text{or} \qquad A = -C = -2$$

Substituting the values $A = -2$, $B = -2$, and $C = 2$

$$\frac{2x+4}{x^2(x-2)} = \frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2}$$

$$\int \frac{2x+4}{x^2(x-2)}\,dx = -2\int \frac{dx}{x} - 2\int \frac{dx}{x^2} + 2\int \frac{dx}{x-2}$$

$$= -2\ln|x| + \frac{2}{x} + 2\ln|x-2| + C = 2\ln\left|\frac{x-2}{x}\right| + \frac{2}{x} + C$$

B. Quadratic factors

Quadratic factor rule

For each factor of the form $(ax^2 + bx + c)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}$$

where $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m$ are constants to be determined. In the case where $m = 1$, only the first term in the sum appears.

Example 2.17: Evaluate

$$\int \frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1}\,dx.$$

Solution. The denominator in the integrand can be factored by grouping:

$$3x^3 - x^2 + 3x - 1 = x^2(3x-1) + (3x-1) = (3x-1)(x^2+1)$$

$$\frac{A}{3x-1}$$

$$\frac{Bx+C}{x^2+1}$$

$$\frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+1}$$

$$x^2+x-2 = A(x^2+1) + (Bx+C)(3x-1)$$

$$x^2+x-2 = (A+3B)x^2 + (-B+3C)x + (A-C)$$

$$\begin{aligned} A + 3B \quad\quad &= 1 \\ -\ B + 3C &= 1 \\ A \quad\quad - C &= -2 \end{aligned}$$

$$A = -\frac{7}{5}, \quad B = \frac{4}{5}, \quad C = \frac{3}{5}$$

$$\frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{-\frac{7}{5}}{3x-1} + \frac{\frac{4}{5}x+\frac{3}{5}}{x^2+1}$$

$$\begin{aligned} \int \frac{x^2+x-2}{(3x-1)(x^2+1)}\,dx &= -\frac{7}{5}\int \frac{dx}{3x-1} + \frac{4}{5}\int \frac{x}{x^2+1}\,dx + \frac{3}{5}\int \frac{dx}{x^2+1} \\ &= -\frac{7}{15}\ln|3x-1| + \frac{2}{5}\ln(x^2+1) + \frac{3}{5}\tan^{-1}x + C \end{aligned}$$

Example 2.18: Evaluate

$$\int \frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}\,dx.$$

Solution:

$$\frac{A}{x+2}$$

$$\frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2}$$

$$\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2} = \frac{A}{x+2} + \frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2}$$

$$\begin{aligned} &3x^4+4x^3+16x^2+20x+9 \\ &\quad = A(x^2+3)^2 + (Bx+C)(x^2+3)(x+2) + (Dx+E)(x+2) \end{aligned}$$

$$\begin{aligned} &3x^4+4x^3+16x^2+20x+9 \\ &\quad = (A+B)x^4 + (2B+C)x^3 + (6A+3B+2C+D)x^2 \\ &\quad\quad + (6B+3C+2D+E)x + (9A+6C+2E) \end{aligned}$$

$$\begin{aligned}A + B &= \;\,3\\2B + C &= \;\,4\\6A + 3B + 2C + D &= 16\\6B + 3C + 2D + E &= 20\\9A + 6C + 2E &= \;\,9\end{aligned}$$

$$\begin{aligned}B &= \;\,2\\2B + C &= \;\,4\\3B + 2C + D &= 10\\6B + 3C + 2D + E &= 20\\6C + 2E &= \;\,0\end{aligned}$$

$$A = 1, \quad B = 2, \quad C = 0, \quad D = 4, \quad E = 0$$

$$\frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2} = \frac{1}{x+2} + \frac{2x}{x^2+3} + \frac{4x}{(x^2+3)^2}$$

$$\begin{aligned}&\int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2}\,dx\\&\qquad = \int \frac{dx}{x+2} + \int \frac{2x}{x^2+3}\,dx + 4\int \frac{x}{(x^2+3)^2}\,dx\\&\qquad = \ln|x+2| + \ln(x^2+3) - \frac{2}{x^2+3} + C\end{aligned}$$

2.6 IMPROPER INTEGRALS

The definite integral

$$\int_a^b f(x)dx$$

that $[a, b]$ is a finite interval and that the limit that defines the integral exists; that is, the function f is integrable.

Extending the concept of a definite integral to allow for infinite intervals of integration and integrands with vertical asymptotes within the interval of integration. The vertical asymptotes is called ***infinite discontinuities***, and integrals with infinite intervals of integration or infinite discontinuities within the interval of integration is called ***improper integrals***. Here are some examples:

- Improper integrals with infinite intervals of integration:

$$\int_1^{+\infty} \frac{dx}{x^2}, \quad \int_{-\infty}^{0} e^x\,dx, \quad \int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$$

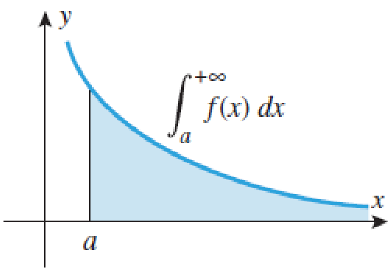
- Improper integrals with infinite discontinuities in the interval of integration:

$$\int_{-3}^{3} \frac{dx}{x^2}, \quad \int_1^2 \frac{dx}{x-1}, \quad \int_0^{\pi} \tan x\,dx$$

- Improper integrals with infinite discontinuities and infinite intervals of integration:

$$\int_0^{+\infty} \frac{dx}{\sqrt{x}}, \quad \int_{-\infty}^{+\infty} \frac{dx}{x^2-9}, \quad \int_1^{+\infty} \sec x\,dx$$

2.6.1 Integrals over Infinite Intervals



Definition The ***improper integral of f over the interval* [*a*, +∞)** is defined to be

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty} \int_a^b f(x)\,dx$$

In the case where the limit exists, the improper integral is said to ***converge***, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to ***diverge***, and it is not assigned a value.

Example 2.19: Evaluate

$$\text{(a)} \int_1^{+\infty} \frac{dx}{x^3} \qquad \text{(b)} \int_1^{+\infty} \frac{dx}{x}$$

Solution (a): Following the definition, we replace the infinite upper limit by a finite upper limit b, and then take the limit of the resulting integral. This yields

$$\int_1^{+\infty} \frac{dx}{x^3} = \lim_{b\to+\infty} \int_1^{b} \frac{dx}{x^3} = \lim_{b\to+\infty} \left[-\frac{1}{2x^2}\right]_1^b = \lim_{b\to+\infty} \left(\frac{1}{2} - \frac{1}{2b^2}\right) = \frac{1}{2}$$

Since the limit is finite, the integral converges and its value is 1/2.

Solution (b):

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{b\to+\infty} \int_1^{b} \frac{dx}{x} = \lim_{b\to+\infty} [\ln x]_1^b = \lim_{b\to+\infty} \ln b = +\infty$$

In this case the integral diverges and hence has no value.

Example 2.20: For what values of p does the integral $\int_1^{+\infty} \frac{dx}{x^p}$ converge?

Solution. We know from the preceding example that the integral diverges if $p = 1$, so let us assume that $p \neq 1$. In this case we have

$$\int_1^{+\infty} \frac{dx}{x^p} = \lim_{b\to+\infty} \int_1^{b} x^{-p}\, dx = \lim_{b\to+\infty} \left.\frac{x^{1-p}}{1-p}\right]_1^b = \lim_{b\to+\infty} \left[\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\right]$$

If $p > 1$, then the exponent $1 - p$ is negative and $b^{1-p} \to 0$ as $b \to +\infty$; and if $p < 1$, then the exponent $1 - p$ is positive and $b^{1-p} \to +\infty$ as $b \to +\infty$. Thus, the integral converges if $p > 1$ and diverges otherwise. In the convergent case the value of the integral is

$$\int_1^{+\infty} \frac{dx}{x^p} = \left[0 - \frac{1}{1-p}\right] = \frac{1}{p-1} \qquad (p > 1)$$

The following theorem summarizes this result.

THEOREM

$$\int_1^{+\infty} \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1} & \text{if} \quad p > 1 \\ \text{diverges} & \text{if} \quad p \leq 1 \end{cases}$$

Example 2.21 Evaluate

$$\int_0^{+\infty} (1-x)e^{-x}\,dx.$$

Solution. We begin by evaluating the indefinite integral using integration by parts. Setting $u = 1 - x$ and $dv = e^{-x}\,dx$ yields

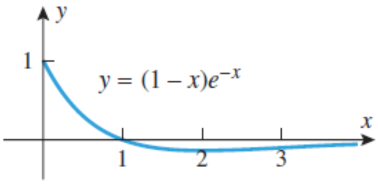
$$\int (1-x)e^{-x}\,dx = -e^{-x}(1-x) - \int e^{-x}\,dx = -e^{-x} + xe^{-x} + e^{-x} + C = xe^{-x} + C$$

$$\int_0^{+\infty} (1-x)e^{-x}\,dx = \lim_{b\to+\infty} \int_0^b (1-x)e^{-x}\,dx = \lim_{b\to+\infty} \left[xe^{-x}\right]_0^b = \lim_{b\to+\infty} \frac{b}{e^b}$$

The limit is an indeterminate form of type ∞/∞, so we will apply L'Hôpital's rule by differentiating the numerator and denominator with respect to b. This yields

$$\int_0^{+\infty} (1-x)e^{-x}\,dx = \lim_{b\to+\infty} \frac{1}{e^b} = 0$$

We can interpret this to mean that the net signed area between the graph of $y = (1 - x)e^{-x}$ and the interval $[0, +\infty)$ is 0 (Figure).



The net signed area between the graph and the interval $[0, +\infty)$ is zero.

Definition The ***improper integral of f over the interval (−∞, b]*** is defined to be

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

The integral is said to ***converge*** if the limit exists and ***diverge*** if it does not.

The ***improper integral of f over the interval (−∞, +∞)*** is defined as

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{+\infty} f(x)\,dx$$

where c is any real number. The improper integral is said to ***converge*** if *both* terms converge and ***diverge*** if *either* term diverges.

Example 2.22: Evaluate

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}.$$

Solution. We will evaluate the integral by choosing $c = 0$. With this value for c we obtain

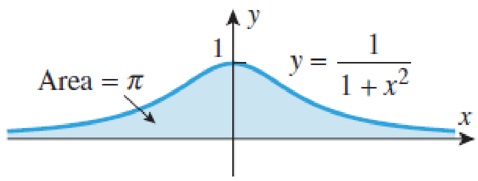
$$\int_{0}^{+\infty} \frac{dx}{1+x^2} = \lim_{b\to+\infty} \int_{0}^{b} \frac{dx}{1+x^2} = \lim_{b\to+\infty} \left[\tan^{-1} x\right]_0^b = \lim_{b\to+\infty} (\tan^{-1} b) = \frac{\pi}{2}$$

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \left[\tan^{-1} x\right]_a^0 = \lim_{a\to-\infty} (-\tan^{-1} a) = \frac{\pi}{2}$$

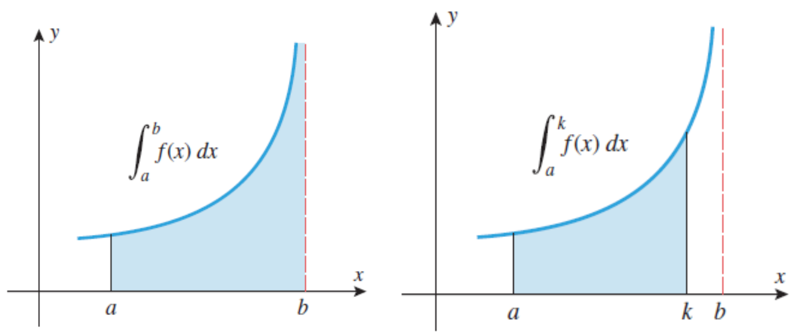
Thus, the integral converges and its value is

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Since the integrand is nonnegative on the interval $(-\infty, +\infty)$, the integral represents the area of the region shown in Figure



2.6.2 Integrals Whose Integrands Have Infinite Discontinuities



Definition If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at b, then the ***improper integral of f over the interval* [a, b]** is defined as

$$\int_{a}^{b} f(x)\,dx = \lim_{k\to b^-} \int_{a}^{k} f(x)\,dx$$

In the case where the indicated limit exists, the improper integral is said to ***converge***, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to ***diverge***, and it is not assigned a value.

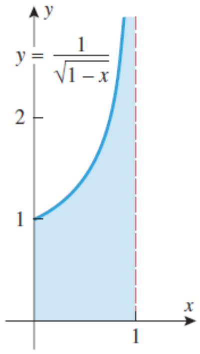
Example 2.23: Evaluate

$$\int_0^1 \frac{dx}{\sqrt{1-x}}.$$

Solution. The integral is improper because the integrand approaches $+\infty$ as x approaches the upper limit 1 from the left (Figure).

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{k\to 1^-}\int_0^k \frac{dx}{\sqrt{1-x}} = \lim_{k\to 1^-}\left[-2\sqrt{1-x}\right]_0^k$$

$$= \lim_{k\to 1^-}\left[-2\sqrt{1-k}+2\right] = 2$$



Definition If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a, then the ***improper integral of f over the interval* [*a*, *b*]** is defined as

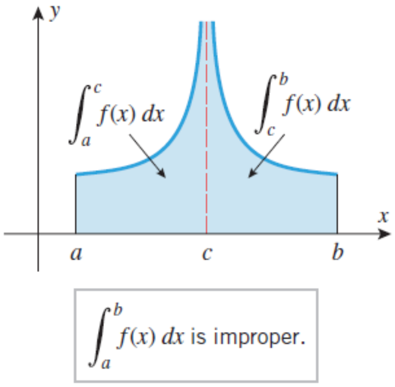
$$\int_a^b f(x)\,dx = \lim_{k\to a^+}\int_k^b f(x)\,dx$$

The integral is said to ***converge*** if the indicated limit exists and ***diverge*** if it does not.

If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a point c in (a, b), then the ***improper integral of f over the interval* [*a*, *b*]** is defined as

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

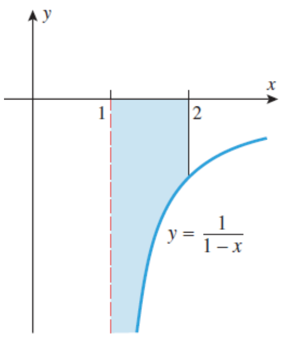
where the two integrals on the right side are themselves improper. The improper integral on the left side is said to ***converge*** if *both* terms on the right side converge and ***diverge*** if *either* term on the right side diverges (Figure).



Example 2.24: Evaluate

$$\text{(a)} \int_1^2 \frac{dx}{1-x} \qquad \text{(b)} \int_1^4 \frac{dx}{(x-2)^{2/3}}$$

Solution (a). The integral is improper because the integrand approaches $-\infty$ as x approaches the lower limit 1 from the right (Figure 7.8.10). From Definition 7.8.5 we obtain



$$\int_1^2 \frac{dx}{1-x} = \lim_{k \to 1^+} \int_k^2 \frac{dx}{1-x} = \lim_{k \to 1^+} \Big[-\ln|1-x|\Big]_k^2$$

$$= \lim_{k \to 1^+} \big[-\ln|-1| + \ln|1-k|\big] = \lim_{k \to 1^+} \ln|1-k| = -\infty$$

Solution (b). The integral is improper because the integrand approaches $+\infty$ at $x = 2$, which is inside the interval of integration. From Definition 7.8.5 we obtain

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}}$$

and we must investigate the convergence of both improper integrals on the right. Since

$$\int_1^2 \frac{dx}{(x-2)^{2/3}} = \lim_{k \to 2^-} \int_1^k \frac{dx}{(x-2)^{2/3}} = \lim_{k \to 2^-} \Big[3(k-2)^{1/3} - 3(1-2)^{1/3}\Big] = 3$$

$$\int_2^4 \frac{dx}{(x-2)^{2/3}} = \lim_{k \to 2^+} \int_k^4 \frac{dx}{(x-2)^{2/3}} = \lim_{k \to 2^+} \Big[3(4-2)^{1/3} - 3(k-2)^{1/3}\Big] = 3\sqrt[3]{2}$$

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = 3 + 3\sqrt[3]{2}$$

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CALCULUS II

Chapter Three

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CHAPTER THREE

INFINITE SERIES

3.1 SEQUENCES

3.1.1 Definition of a Sequence

Stated informally, an ***infinite sequence***, or more simply a ***sequence***, is an unending succession of numbers, called ***terms***. It is understood that the terms have a definite order; that is, there is a first term a_1, a second term a_2, a third term a_3, a fourth term a_4, and so forth. Such a sequence would typically be written as

$$a_1, a_2, a_3, a_4, \ldots$$

$1, 2, 3, 4, \ldots,$ $\qquad$ $1, 1/2, 1/3, 1/4, \ldots,$

$2, 4, 6, 8, \ldots,$ $\qquad$ $1, -1, 1, -1, \ldots$

Example 1 In each part, find the general term of the sequence.

(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ $\qquad$ (b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

(c) $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots$ $\qquad$ (d) $1, 3, 5, 7, \ldots$

Solution (a).

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$$

TERM NUMBER	1	2	3	4	⋯	n	⋯
TERM	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	⋯	$\frac{n}{n+1}$	⋯

Solution (b).

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$$

TERM NUMBER	1	2	3	4	⋯	n	⋯
TERM	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	⋯	$\frac{1}{2^n}$	⋯

Solution (c).

$$\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots, (-1)^{n+1}\frac{n}{n+1}, \ldots$$

Solution (d).

$$1, 3, 5, 7, \ldots, 2n-1, \ldots$$

TERM NUMBER	1	2	3	4	⋯	n	⋯
TERM	1	3	5	7	⋯	$2n-1$	⋯

SEQUENCE	BRACE NOTATION
$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$	$\left\{\frac{n}{n+1}\right\}_{n=1}^{+\infty}$
$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$	$\left\{\frac{1}{2^n}\right\}_{n=1}^{+\infty}$
$\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots, (-1)^{n+1}\frac{n}{n+1}, \ldots$	$\left\{(-1)^{n+1}\frac{n}{n+1}\right\}_{n=1}^{+\infty}$
$1, 3, 5, 7, \ldots, 2n-1, \ldots$	$\{2n-1\}_{n=1}^{+\infty}$

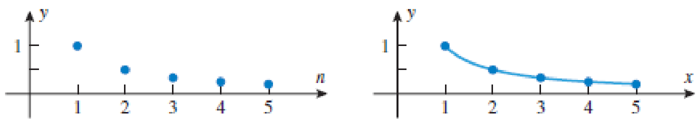
Definition A ***sequence*** is a function whose domain is a set of integers.

3.1.2 Graphs of Sequences

For example, the graph of the sequence $\{1/n\}_{n=1}^{+\infty}$ is the graph of the equation

$$y = \frac{1}{n} \qquad n = 1, 2, 3, \ldots..$$

$$y = \frac{1}{x} \qquad x \geq 1$$

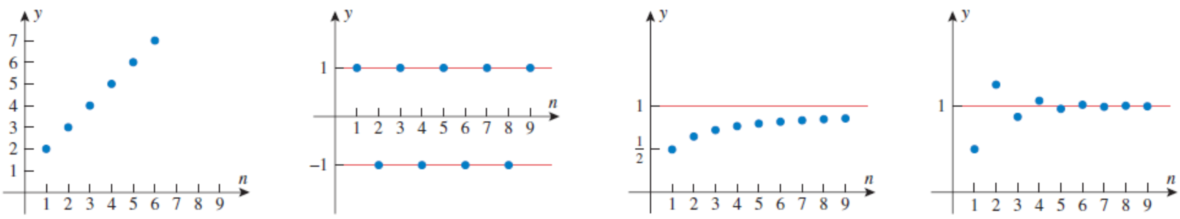


$y = \frac{1}{n},\ n = 1, 2, 3, \ldots$ (*a*) $y = \frac{1}{x},\ x \geq 1$ (*b*)

Figure 1

3.1.3 Limit of a Sequence

- The terms in the sequence $\{n+1\}$ increase without bound.
- The terms in the sequence $\{(-1)^{n+1}\}$ oscillate between -1 and 1.
- The terms in the sequence $\{n/(n+1)\}$ increase toward a "limiting value" of 1.
- The terms in the sequence $\left\{1 + \left(-\frac{1}{2}\right)^n\right\}$ also tend toward a "limiting value" of 1, but do so in an oscillatory fashion.



$\{n+1\}_{n=1}^{+\infty}$ $\{(-1)^{n+1}\}_{n=1}^{+\infty}$ $\left\{\frac{n}{n+1}\right\}_{n=1}^{+\infty}$ $\left\{1 + \left(-\frac{1}{2}\right)^n\right\}_{n=1}^{+\infty}$

Figure 2

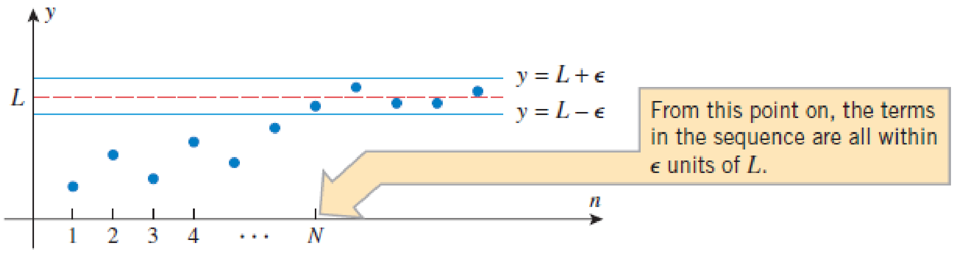


Figure 3

DEFINITION A sequence $\{a_n\}$ is said to ***converge*** to the ***limit*** L if given any $\epsilon > 0$, there is a positive integer N such that $|a_n - L| < \epsilon$ for $n \geq N$. In this case we write

$$\lim_{n\to+\infty} a_n = L$$

A sequence that does not converge to some finite limit is said to ***diverge***.

Example 2 The first two sequences in Figure 2 diverge, and the second two converge to 1; that is,

$$\lim_{n\to+\infty} \frac{n}{n+1} = 1 \quad \text{and} \quad \lim_{n\to+\infty} \left[1 + \left(-\tfrac{1}{2}\right)^n\right] = 1$$

THEOREM *Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ converge to limits L_1 and L_2, respectively, and c is a constant. Then:*

(a) $\displaystyle \lim_{n\to+\infty} c = c$

(b) $\displaystyle \lim_{n\to+\infty} ca_n = c \lim_{n\to+\infty} a_n = cL_1$

(c) $\displaystyle \lim_{n\to+\infty} (a_n + b_n) = \lim_{n\to+\infty} a_n + \lim_{n\to+\infty} b_n = L_1 + L_2$

(d) $\displaystyle \lim_{n\to+\infty} (a_n - b_n) = \lim_{n\to+\infty} a_n - \lim_{n\to+\infty} b_n = L_1 - L_2$

(e) $\displaystyle \lim_{n\to+\infty} (a_n b_n) = \lim_{n\to+\infty} a_n \cdot \lim_{n\to+\infty} b_n = L_1 L_2$

(f) $\displaystyle \lim_{n\to+\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to+\infty} a_n}{\lim_{n\to+\infty} b_n} = \frac{L_1}{L_2} \quad (\text{if } L_2 \neq 0)$

Example 3 In each part, determine whether the sequence converges or diverges by examining the limit as n→+∞.

$$\text{(a)}\ \left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty} \qquad \text{(b)}\ \left\{(-1)^{n+1}\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$$

$$\text{(c)}\ \left\{(-1)^{n+1}\frac{1}{n}\right\}_{n=1}^{+\infty} \qquad \text{(d)}\ \{8 - 2n\}_{n=1}^{+\infty}$$

Solution (a). Dividing numerator and denominator by n and using Theorem yields

$$\lim_{n\to+\infty}\frac{n}{2n+1}=\lim_{n\to+\infty}\frac{1}{2+1/n}=\frac{\lim\limits_{n\to+\infty}1}{\lim\limits_{n\to+\infty}(2+1/n)}=\frac{\lim\limits_{n\to+\infty}1}{\lim\limits_{n\to+\infty}2+\lim\limits_{n\to+\infty}1/n}$$
$$=\frac{1}{2+0}=\frac{1}{2}$$

Thus, the sequence converges to 1/2.

Solution (b). This sequence is the same as that in part (a), except for the factor of $(-1)^{n+1}$, which oscillates between +1 and −1. Thus, the terms in this sequence oscillate between positive and negative values, with the odd-numbered terms being identical to those in part (a) and the even-numbered terms being the negatives of those in part (a). Since the sequence in part (a) has a limit of 1/2, it follows that the odd-numbered terms in this sequence approach 1/2, and the even-numbered terms approach −1/2. Therefore, this sequence has no limit—it diverges.

Solution (c). Since $(1/n)\to 0$, the product $(-1)^{n+1}(1/n)$ oscillates between positive and negative values, with the odd-numbered terms approaching 0 through positive values and the even-numbered terms approaching 0 through negative values. Thus,

$$\lim_{n\to+\infty}(-1)^{n+1}\frac{1}{n}=0$$

so the sequence converges to 0.

Solution (d).

$$\lim_{n\to+\infty}(8-2n)=-\infty,\text{ so the sequence }\{8-2n\}_{n=1}^{+\infty}\text{ diverges.}$$

Example 4 In each part, determine whether the sequence converges, and if so, find its limit.

$$\text{(a)}\ \ 1,\frac{1}{2},\frac{1}{2^2},\frac{1}{2^3},\ldots,\frac{1}{2^n},\ldots\qquad\text{(b)}\ \ 1,2,2^2,2^3,\ldots,2^n,\ldots$$

Solution. Replacing n by x in the first sequence produces the power function $(1/2)^x$, and replacing n by x in the second sequence produces the power function 2^x . Now recall that if $0 < b < 1$, then $b^x\to 0$ as $x\to+\infty$, and if $b > 1$, then $b^x\to+\infty$ as $x\to+\infty$ (Figure 1).

Thus,

$$\lim_{n\to+\infty}\frac{1}{2^n}=0\quad\text{and}\quad\lim_{n\to+\infty}2^n=+\infty$$

So, the sequence $\{1/2^n\}$ converges to 0, but the sequence $\{2^n\}$ diverges.

Example 5 Find the limit of the sequence

$$\left\{\frac{n}{e^n}\right\}_{n=1}^{+\infty}.$$

Solution. The expression

$$\lim_{n\to+\infty}\frac{n}{e^n}$$

is an indeterminate form of type ∞/∞, so L'Hôpital's rule is indicated. However, we cannot apply this rule directly to n/e^n because the functions n and e^n have been defined here only at the positive integers, and hence are not differentiable functions. To circumvent this problem we extend the domains of these functions to all real numbers, here implied by replacing n by x, and apply L'Hôpital's rule to the limit of the quotient x/e^x. This yields

$$\lim_{x\to+\infty}\frac{x}{e^x}=\lim_{x\to+\infty}\frac{1}{e^x}=0$$

from which we can conclude that

$$\lim_{n\to+\infty}\frac{n}{e^n}=0$$

Example 6 Show that

$$\lim_{n\to+\infty}\sqrt[n]{n}=1.$$

Solution.

$$\lim_{n\to+\infty}\sqrt[n]{n}=\lim_{n\to+\infty}n^{1/n}=\lim_{n\to+\infty}e^{(1/n)\ln n}=e^0=1$$

By L'Hôpital's rule applied to $(1/x)\ln x$

Theorem *A sequence converges to a limit L if and only if the sequences of even-numbered terms and odd-numbered terms both converge to L.*

Example 7 The sequence

$$\frac{1}{2},\frac{1}{3},\frac{1}{2^2},\frac{1}{3^2},\frac{1}{2^3},\frac{1}{3^3},\ldots$$

converges to 0, since the even-numbered terms and the odd-numbered terms both converge to 0, and the sequence

$$1,\tfrac{1}{2},1,\tfrac{1}{3},1,\tfrac{1}{4},\ldots$$

diverges, since the odd-numbered terms converge to 1 and the even-numbered terms converge to 0.

3.1.4 The Squeezing Theorem for Sequences

THEOREM (*The Squeezing Theorem for Sequences*) *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences such that*

$$a_n \le b_n \le c_n \quad (\textit{for all values of } n \textit{ beyond some index } N)$$

If the sequences $\{a_n\}$ and $\{c_n\}$ have a common limit L as $n \to +\infty$, then $\{b_n\}$ also has the limit L as $n \to +\infty$.

Example 8 Use numerical evidence to make a conjecture about the limit of the sequence

$$\left\{\frac{n!}{n^n}\right\}_{n=1}^{+\infty}$$

and then confirm that your conjecture is correct.

n	$\frac{n!}{n^n}$
1	1.0000000000
2	0.5000000000
3	0.2222222222
4	0.0937500000
5	0.0384000000
6	0.0154320988
7	0.0061198990
8	0.0024032593
9	0.0009366567
10	0.0003628800
11	0.0001399059
12	0.0000537232

Solution. The following table, which was obtained with a calculating utility, suggests that the limit of the sequence may be 0. To confirm this we need to examine the limit of

$$a_n = \frac{n!}{n^n}$$

As $n \to +\infty$. Although this is an indeterminate form of type ∞/∞, L'Hôpital's rule is not helpful because we have no definition of $x!$ for values of x that are not integers. However, let us write out some of the initial terms and the general term in the sequence:

$$a_1 = 1, \quad a_2 = \frac{1\cdot 2}{2\cdot 2} = \frac{1}{2}, \quad a_3 = \frac{1\cdot 2\cdot 3}{3\cdot 3\cdot 3} = \frac{2}{9} < \frac{1}{3}, \quad a_4 = \frac{1\cdot 2\cdot 3\cdot 4}{4\cdot 4\cdot 4\cdot 4} = \frac{3}{32} < \frac{1}{4}, \ldots$$

If n > 1, the general term of the sequence can be rewritten as

$$a_n = \frac{1\cdot 2\cdot 3\cdots n}{n\cdot n\cdot n\cdots n} = \frac{1}{n}\left(\frac{2\cdot 3\cdots n}{n\cdot n\cdots n}\right)$$

from which it follows that $a_n \le 1/n$ (why?). It is now evident that

$$0 \le a_n \le \frac{1}{n}$$

However, the two outside expressions have a limit of 0 as $n \to +\infty$; thus, the Squeezing Theorem for Sequences implies that $a_n \to 0$ as $n \to +\infty$, which confirms our conjecture.

THEOREM *If* $\lim_{n\to+\infty} |a_n| = 0$, *then* $\lim_{n\to+\infty} a_n = 0$.

Example 9 Consider the sequence

$$1,\ -\frac{1}{2},\ \frac{1}{2^2},\ -\frac{1}{2^3},\ \ldots,\ (-1)^n\frac{1}{2^n},\ \ldots$$

If we take the absolute value of each term, we obtain the sequence

$$1,\ \frac{1}{2},\ \frac{1}{2^2},\ \frac{1}{2^3},\ \ldots,\ \frac{1}{2^n},\ \ldots$$

which, as shown in Example 4, converges to 0. Thus, from Theorem we have

$$\lim_{n\to+\infty}\left[(-1)^n\frac{1}{2^n}\right]=0$$

3.2 MONOTONE SEQUENCES

3.2.1 Terminology

DEFINITION A sequence $\{a_n\}_{n=1}^{+\infty}$ is called

strictly increasing if	$a_1 < a_2 < a_3 < \cdots < a_n < \cdots$
increasing if	$a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots$
strictly decreasing if	$a_1 > a_2 > a_3 > \cdots > a_n > \cdots$
decreasing if	$a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n \ge \cdots$

A sequence that is either increasing or decreasing is said to be ***monotone***, and a sequence that is either strictly increasing or strictly decreasing is said to be ***strictly monotone***.

Some examples are given in the below table

SEQUENCE	DESCRIPTION
$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$	Strictly increasing
$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$	Strictly decreasing
$1, 1, 2, 2, 3, 3, \ldots$	Increasing; not strictly increasing
$1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \ldots$	Decreasing; not strictly decreasing
$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots$	Neither increasing nor decreasing

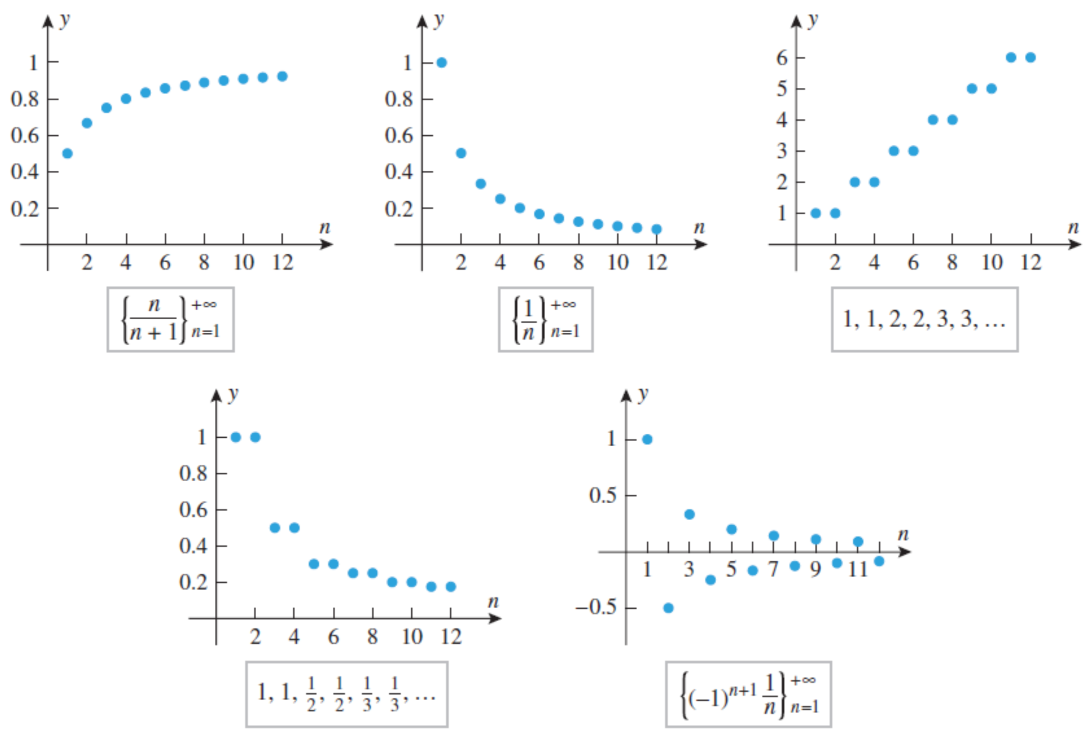


Figure 4

3.2.2 Testing for Monotonicity

Frequently, one can *guess* whether a sequence is monotone or strictly monotone by writing out some of the initial terms.

However, to be certain that the guess is correct, one must give a precise mathematical argument. The below table provides two ways of doing this, one based on differences of successive terms and the other on ratios of successive terms.

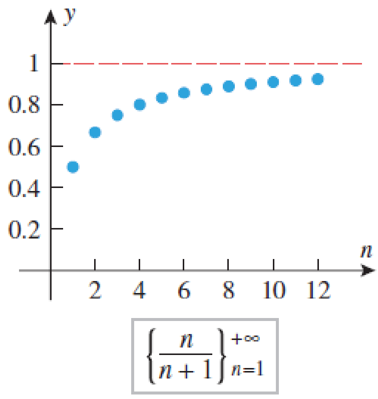
DIFFERENCE BETWEEN SUCCESSIVE TERMS	RATIO OF SUCCESSIVE TERMS	CONCLUSION
$a_{n+1} - a_n > 0$	$a_{n+1}/a_n > 1$	Strictly increasing
$a_{n+1} - a_n < 0$	$a_{n+1}/a_n < 1$	Strictly decreasing
$a_{n+1} - a_n \geq 0$	$a_{n+1}/a_n \geq 1$	Increasing
$a_{n+1} - a_n \leq 0$	$a_{n+1}/a_n \leq 1$	Decreasing

Example 10 Use differences of successive terms to show that

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$$

(Figure 4) is a strictly increasing sequence.

Solution. The pattern of the initial terms suggests that the sequence is strictly increasing. To prove that this is so, let



$$a_n = \frac{n}{n+1}$$

We can obtain a_{n+1} by replacing n by $n + 1$ in this formula. This yields

$$a_{n+1} = \frac{n+1}{(n+1)+1} = \frac{n+1}{n+2}$$

Thus, for $n \geq 1$

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{n^2+2n+1-n^2-2n}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} > 0$$

which proves that the sequence is strictly increasing.

Example 11 Use ratios of successive terms to show that the sequence in Example 10 is strictly increasing.

Solution. As shown in the solution of Example 10,

$$a_n = \frac{n}{n+1} \quad \text{and} \quad a_{n+1} = \frac{n+1}{n+2}$$

Forming the ratio of successive terms we obtain

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)/(n+2)}{n/(n+1)} = \frac{n+1}{n+2} \cdot \frac{n+1}{n} = \frac{n^2+2n+1}{n^2+2n}$$

from which we see that $a_{n+1}/a_n > 1$ for $n \geq 1$. This proves that the sequence is strictly increasing.

Example 12 In Examples 10 and 11 we proved that the sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$$

is strictly increasing by considering the difference and ratio of successive terms. Alternatively, we can proceed as follows. Let

$$f(x) = \frac{x}{x+1}$$

so that the nth term in the given sequence is $a_n = f(n)$. The function f is increasing for $x \geq 1$ since

$$f'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2} > 0$$

Thus, $a_n = f(n) < f(n + 1) = a_{n+1}$

which proves that the given sequence is strictly increasing.

DERIVATIVE OF f FOR $x \geq 1$	CONCLUSION FOR THE SEQUENCE WITH $a_n = f(n)$
$f'(x) > 0$	Strictly increasing
$f'(x) < 0$	Strictly decreasing
$f'(x) \geq 0$	Increasing
$f'(x) \leq 0$	Decreasing

In general, if $f(n) = a_n$ is the nth term of a sequence, and if f is differentiable for $x \geq 1$, then the results in Table can be used to investigate the monotonicity of the sequence.

3.2.3 Properties That Hold Eventually

Definition If discarding finitely many terms from the beginning of a sequence produces a sequence with a certain property, then the original sequence is said to have that property ***eventually***.

Example 13 Show that the sequence $\left\{\frac{10^n}{n!}\right\}_{n=1}^{+\infty}$ is eventually strictly decreasing.

Solution. We have

$$a_n = \frac{10^n}{n!} \quad \text{and} \quad a_{n+1} = \frac{10^{n+1}}{(n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{10^{n+1}/(n+1)!}{10^n/n!} = \frac{10^{n+1}n!}{10^n(n+1)!} = 10\frac{n!}{(n+1)n!} = \frac{10}{n+1} \qquad \text{(A)}$$

From (A), $a_{n+1}/a_n < 1$ for all $n \geq 10$, so the sequence is eventually strictly decreasing, as confirmed by the graph in Figure 5.

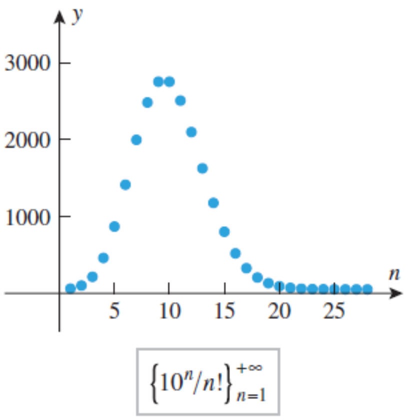


Figure 5

3.2.4 Convergence of Monotone Sequences

THEOREM *If a sequence $\{a_n\}$ is eventually increasing, then there are two possibilities:*

(*a*) *There is a constant M, called an **upper bound** for the sequence, such that $a_n \leq M$ for all n, in which case the sequence converges to a limit L satisfying $L \leq M$.*

(*b*) *No upper bound exists, in which case $\lim_{n\to+\infty} a_n = +\infty$.*

THEOREM *If a sequence $\{a_n\}$ is eventually decreasing, then there are two possibilities:*

(a) *There is a constant M, called a **lower bound** for the sequence, such that $a_n \geq M$ for all n, in which case the sequence converges to a limit L satisfying $L \geq M$.*

(b) *No lower bound exists, in which case $\lim_{n\to+\infty} a_n = -\infty$.*

Example 14 Show that the sequence $\left\{\frac{10^n}{n!}\right\}_{n=1}^{+\infty}$ converges and find its limit.

Solution. We showed in Example 13 that the sequence is eventually strictly decreasing. Since all terms in the sequence are positive, it is bounded below by $M = 0$, and hence Theorem guarantees that it converges to a nonnegative limit L. However, the limit is not evident directly from the formula $10^n/n!$ for the nth term, so we will need some ingenuity to obtain it.

It follows from Formula (A) of Example 13 that successive terms in the given sequence are related by the recursion formula

$$a_{n+1} = \frac{10}{n+1}a_n$$

where $a_n = 10^n/n!$. We will take the limit as $n\to+\infty$ of both sides and use the fact that

$$\lim_{n\to+\infty} a_{n+1} = \lim_{n\to+\infty} a_n = L$$

We obtain

$$L = \lim_{n\to+\infty} a_{n+1} = \lim_{n\to+\infty}\left(\frac{10}{n+1}a_n\right) = \lim_{n\to+\infty}\frac{10}{n+1}\lim_{n\to+\infty} a_n = 0\cdot L = 0$$

so that

$$L = \lim_{n\to+\infty}\frac{10^n}{n!} = 0$$

In the exercises we will show that the technique illustrated in the last example can be adapted to obtain

$$\boxed{\lim_{n\to+\infty}\frac{x^n}{n!} = 0}$$

3.3 INFINITE SERIES

3.3.1 Sums of Infinite Series

Definition An ***infinite series*** is an expression that can be written in the form

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots$$

The numbers $u_1, u_2, u_3, \ldots$ are called the ***terms*** of the series.

Example of infinite series

$$0.3 + 0.03 + 0.003 + 0.0003 + \cdots$$

or, equivalently,

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots$$

$$\begin{aligned}
s_1 &= \frac{3}{10} = 0.3 \\
s_2 &= \frac{3}{10} + \frac{3}{10^2} = 0.33 \\
s_3 &= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} = 0.333 \\
s_4 &= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} = 0.3333 \\
&\vdots
\end{aligned}$$

$$s_n = \frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^n}$$

$$\begin{aligned}
s_1 &= u_1 \\
s_2 &= u_1 + u_2 \\
s_3 &= u_1 + u_2 + u_3 \\
&\vdots \\
s_n &= u_1 + u_2 + u_3 + \cdots + u_n = \sum_{k=1}^{n} u_k
\end{aligned}$$

The number s_n is called the ***nth partial sum*** of the series and the sequence $\{s_n\}^{+\infty}{}_{n=1}$ is called the ***sequence of partial sums***.

Definition Let $\{s_n\}$ be the sequence of partial sums of the series

$$u_1 + u_2 + u_3 + \cdots + u_k + \cdots$$

If the sequence $\{s_n\}$ converges to a limit S, then the series is said to ***converge*** to S, and S is called the ***sum*** of the series. We denote this by writing

$$S = \sum_{k=1}^{\infty} u_k$$

If the sequence of partial sums diverges, then the series is said to ***diverge***. A divergent series has no sum.

Example 15 Determine whether the series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

converges or diverges. If it converges, find the sum.

Solution.

$s_1 = 1$

$s_2 = 1 - 1 = 0$

$s_3 = 1 - 1 + 1 = 1$

$s_4 = 1 - 1 + 1 - 1 = 0$

and so forth. Thus, the sequence of partial sums is

1, 0, 1, 0, 1, 0, . . .

(Figure 6). Since this is a divergent sequence, the given series diverges and consequently has no sum.

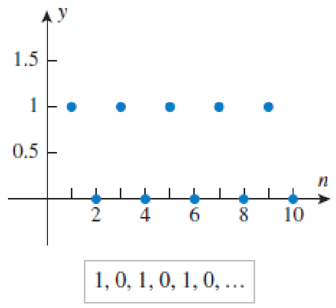


Figure 6

3.3.2 Geometric Series

THEOREM *A geometric series*

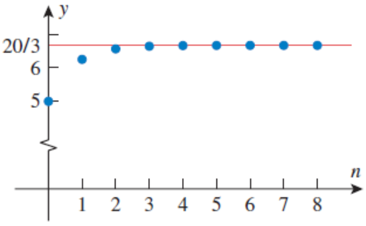
$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \cdots + ar^k + \cdots \quad (a \neq 0)$$

converges if $|r| < 1$ *and diverges if* $|r| \geq 1$. *If the series converges, then the sum is*

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

Example 16 In each part, determine whether the series converges, and if so find its sum.

$$\text{(a)} \sum_{k=0}^{\infty} \frac{5}{4^k} \qquad \text{(b)} \sum_{k=1}^{\infty} 3^{2k} 5^{1-k}$$



Partial sums for $\sum_{k=0}^{\infty} \frac{5}{4^k}$

Solution (a). This is a geometric series with $a = 5$ and $r = 1/4$. Since $|r| = 1/4 < 1$, the series converges and the sum is

$$\frac{a}{1-r} = \frac{5}{1-\frac{1}{4}} = \frac{20}{3}$$

Solution (b). This is a geometric series in concealed form, since we can rewrite it as

$$\sum_{k=1}^{\infty} 3^{2k}5^{1-k} = \sum_{k=1}^{\infty} \frac{9^k}{5^{k-1}} = \sum_{k=1}^{\infty} 9\left(\frac{9}{5}\right)^{k-1}$$

Since $r = 9/5 > 1$, the series diverges.

Example 17 Find the rational number represented by the repeating decimal

$$0.784784784\ldots$$

Solution. We can write

$$0.784784784\ldots = 0.784 + 0.000784 + 0.000000784 + \cdots$$

so the given decimal is the sum of a geometric series with $a = 0.784$ and $r = 0.001$. Thus,

$$0.784784784\ldots = \frac{a}{1-r} = \frac{0.784}{1-0.001} = \frac{0.784}{0.999} = \frac{784}{999}$$

Example 18 In each part, find all values of x for which the series converges, and find the sum of the series for those values of x.

$$\text{(a)}\ \sum_{k=0}^{\infty} x^k \qquad \text{(b)}\ 3 - \frac{3x}{2} + \frac{3x^2}{4} - \frac{3x^3}{8} + \cdots + \frac{3(-1)^k}{2^k}x^k + \cdots$$

Solution (a). The expanded form of the series is

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots + x^k + \cdots$$

The series is a geometric series with $a = 1$ and $r = x$, so it converges if $|x| < 1$ and diverges otherwise. When the series converges its sum is

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

Solution (b). This is a geometric series with $a = 3$ and $r = -x/2$. It converges if $|-x/2| < 1$, or equivalently, when $|x| < 2$. When the series converges its sum is

$$\sum_{k=0}^{\infty} 3\left(-\frac{x}{2}\right)^k = \frac{3}{1-\left(-\frac{x}{2}\right)} = \frac{6}{2+x}$$

3.3.3 Telescoping Sums

Example 19 Determine whether the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots$$

converges or diverges. If it converges, find the sum.

Solution. The nth partial sum of the series is

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$$

We will begin by rewriting s_n in closed form. This can be accomplished by using the method of partial fractions to obtain (verify).

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

from which we obtain the sum

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{n} + \frac{1}{n}\right) - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n \to +\infty} s_n = \lim_{n \to +\infty} \left(1 - \frac{1}{n+1}\right) = 1$$

3.3.4 Harmonic Series

One of the most important of all diverging series is the ***harmonic series***,

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

which arises in connection with the overtones produced by a vibrating musical string. It is not immediately evident that this series diverges. However, the divergence will become apparent when we examine the partial sums in detail. Because the terms in the series are all positive, the partial sums

$$s_1 = 1, \quad s_2 = 1 + \frac{1}{2}, \quad s_3 = 1 + \frac{1}{2} + \frac{1}{3}, \quad s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \ldots$$

form a strictly increasing sequence

$$s_1 < s_2 < s_3 < \cdots < s_n < \cdots$$

$$
\begin{aligned}
s_2 &= 1 + \tfrac{1}{2} > \tfrac{1}{2} + \tfrac{1}{2} = \tfrac{2}{2} \\
s_4 &= s_2 + \tfrac{1}{3} + \tfrac{1}{4} > s_2 + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) = s_2 + \tfrac{1}{2} > \tfrac{3}{2} \\
s_8 &= s_4 + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} > s_4 + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right) = s_4 + \tfrac{1}{2} > \tfrac{4}{2} \\
s_{16} &= s_8 + \tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} + \tfrac{1}{12} + \tfrac{1}{13} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{16} \\
&\quad > s_8 + \left(\tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16}\right) = s_8 + \tfrac{1}{2} > \tfrac{5}{2} \\
&\vdots \\
s_{2^n} &> \frac{n+1}{2}
\end{aligned}
$$

3.4 CONVERGENCE TESTS

3.4.1 The Divergence Test

THEOREM ***(The Divergence Test)***

(*a*) *If* $\lim\limits_{k\to+\infty} u_k \neq 0$, *then the series* $\sum u_k$ *diverges.*

(*b*) *If* $\lim\limits_{k\to+\infty} u_k = 0$, *then the series* $\sum u_k$ *may either converge or diverge.*

THEOREM *If the series* $\sum u_k$ *converges, then* $\lim\limits_{k\to+\infty} u_k = 0$.

Example 20 The series

$$\sum_{k=1}^{\infty} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{k}{k+1} + \cdots$$

diverges since

$$\lim_{k\to+\infty} \frac{k}{k+1} = \lim_{k\to+\infty} \frac{1}{1+1/k} = 1 \neq 0$$

3.4.2 Algebraic Properties of Infinite Series

THEOREM

(*a*) *If* $\sum u_k$ *and* $\sum v_k$ *are convergent series, then* $\sum(u_k + v_k)$ *and* $\sum(u_k - v_k)$ *are convergent series and the sums of these series are related by*

$$\sum_{k=1}^{\infty}(u_k + v_k) = \sum_{k=1}^{\infty} u_k + \sum_{k=1}^{\infty} v_k$$

$$\sum_{k=1}^{\infty}(u_k - v_k) = \sum_{k=1}^{\infty} u_k - \sum_{k=1}^{\infty} v_k$$

(*b*) *If c is a nonzero constant, then the series* $\sum u_k$ *and* $\sum cu_k$ *both converge or both diverge. In the case of convergence, the sums are related by*

$$\sum_{k=1}^{\infty} cu_k = c\sum_{k=1}^{\infty} u_k$$

(*c*) *Convergence or divergence is unaffected by deleting a finite number of terms from a series; in particular, for any positive integer* K, *the series*

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots$$

$$\sum_{k=K}^{\infty} u_k = u_K + u_{K+1} + u_{K+2} + \cdots$$

both converge or both diverge.

Example 21 Find the sum of the series

$$\sum_{k=1}^{\infty}\left(\frac{3}{4^k} - \frac{2}{5^{k-1}}\right)$$

Solution. The series

$$\sum_{k=1}^{\infty}\frac{3}{4^k} = \frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \cdots$$

is a convergent geometric series ($a = 3/4$, $r = 1/4$), and the series

$$\sum_{k=1}^{\infty}\frac{2}{5^{k-1}} = 2 + \frac{2}{5} + \frac{2}{5^2} + \frac{2}{5^3} + \cdots$$

is also a convergent geometric series ($a = 2$, $r = 1/5$). Thus,

$$\sum_{k=1}^{\infty}\left(\frac{3}{4^k}-\frac{2}{5^{k-1}}\right)=\sum_{k=1}^{\infty}\frac{3}{4^k}-\sum_{k=1}^{\infty}\frac{2}{5^{k-1}}$$
$$=\frac{\frac{3}{4}}{1-\frac{1}{4}}-\frac{2}{1-\frac{1}{5}}=-\frac{3}{2}$$

3.4.3 The Integral Test

THEOREM (*The Integral Test*) *Let $\sum u_k$ be a series with positive terms. If f is a function that is decreasing and continuous on an interval $[a, +\infty)$ and such that $u_k = f(k)$ for all $k \geq a$, then*

$$\sum_{k=1}^{\infty} u_k \quad \textit{and} \quad \int_a^{+\infty} f(x)\,dx$$

both converge or both diverge.

Example 22 Show that the integral test applies, and use the integral test to determine whether the following series converge or diverge.

$$\text{(a)} \sum_{k=1}^{\infty}\frac{1}{k} \qquad \text{(b)} \sum_{k=1}^{\infty}\frac{1}{k^2}$$

Solution (a). We already know that this is the divergent harmonic series, so the integral test will simply illustrate another way of establishing the divergence. Note first that the series has positive terms, so the integral test is applicable. If we replace k by x in the general term $1/k$, we obtain the function $f(x) = 1/x$, which is decreasing and continuous for $x \geq 1$ (as required to apply the integral test with $a = 1$). Since

$$\int_1^{+\infty}\frac{1}{x}\,dx=\lim_{b\to+\infty}\int_1^b\frac{1}{x}\,dx=\lim_{b\to+\infty}[\ln b-\ln 1]=+\infty$$

the integral diverges and consequently so does the series.

Solution (b). Note first that the series has positive terms, so the integral test is applicable. If we replace k by x in the general term $1/k^2$, we obtain the function $f(x) = 1/x^2$, which is decreasing and continuous for $x \geq 1$. Since

$$\int_1^{+\infty}\frac{1}{x^2}\,dx=\lim_{b\to+\infty}\int_1^b\frac{dx}{x^2}=\lim_{b\to+\infty}\left[-\frac{1}{x}\right]_1^b=\lim_{b\to+\infty}\left[1-\frac{1}{b}\right]=1$$

the integral converges and consequently the series converges by the integral test with $a = 1$.

3.4.4 *p*-Series

The special cases of a class of series called ***p*-series** or ***hyperharmonic series***. A *p*-series is an infinite series of the form

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{k^p} + \cdots$$

Where $p > 0$. Examples of *p*-series are

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \cdots \qquad \boxed{p = 1}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \cdots \qquad \boxed{p = 2}$$

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \cdots \qquad \boxed{p = \tfrac{1}{2}}$$

THEOREM (*Convergence of p-Series*)

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{k^p} + \cdots$$

converges if $p > 1$ *and diverges if* $0 < p \leq 1$.

Example 23

$$1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \cdots + \frac{1}{\sqrt[3]{k}} + \cdots$$

diverges since it is a *p*-series with $p = 1/3 < 1$.

3.5 THE COMPARISON, RATIO, AND ROOT TESTS

3.5.1 The Comparison Test

THEOREM (*The Comparison Test*) *Let* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *be series with nonnegative terms and suppose that*

$$a_1 \leq b_1,\ a_2 \leq b_2,\ a_3 \leq b_3, \ldots, a_k \leq b_k, \ldots$$

(*a*) *If the "bigger series"* Σb_k *converges, then the "smaller series"* Σa_k *also converges.*

(*b*) *If the "smaller series"* Σa_k *diverges, then the "bigger series"* Σb_k *also diverges.*

There are two steps required for using the comparison test to determine whether a series Σu_k with positive terms converges:

Step 1. Guess at whether the series Σu_k converges or diverges.

Step 2. Find a series that proves the guess to be correct. That is, if we guess that Σu_k diverges, we must find a divergent series whose terms are "smaller" than the corresponding terms of Σu_k, and if we guess that Σu_k converges, we must find a convergent series whose terms are "bigger" than the corresponding terms of Σu_k.

Informal principle (1) *Constant terms in the denominator of* u_k *can usually be deleted without affecting the convergence or divergence of the series.*

Informal principle (2) *If a polynomial in k appears as a factor in the numerator or denominator of* u_k*, all but the leading term in the polynomial can usually be discarded without affecting the convergence or divergence of the series.*

Example 24 Use the comparison test to determine whether the following series converge or diverge.

$$\text{(a)} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k} - \frac{1}{2}} \qquad \text{(b)} \sum_{k=1}^{\infty} \frac{1}{2k^2 + k}$$

Solution (a). According to Principle 1, we should be able to drop the constant in the denominator without affecting the convergence or divergence. Thus, the given series is likely to behave like

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \quad (1)$$

which is a divergent p-series ($p = 1/2$). Thus, we will guess that the given series diverges and try to prove this by finding a divergent series that is "smaller" than the given series. However, series (1) does the trick since

$$\frac{1}{\sqrt{k}-\frac{1}{2}} > \frac{1}{\sqrt{k}} \quad \text{for } k = 1, 2, \ldots$$

Thus, we have proved that the given series diverges.

Solution (b). According to Principle 2, we should be able to discard all but the leading term in the polynomial without affecting the convergence or divergence. Thus, the given series is likely to behave like

$$\sum_{k=1}^{\infty} \frac{1}{2k^2} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^2} \qquad (2)$$

which converges since it is a constant times a convergent p-series ($p = 2$). Thus, we will guess that the given series converges and try to prove this by finding a convergent series that is "bigger" than the given series. However, series (2) does the trick since

$$\frac{1}{2k^2+k} < \frac{1}{2k^2} \quad \text{for } k = 1, 2, \ldots$$

Thus, we have proved that the given series converges.

3.5.2 The Ratio Test

THEOREM (*The Ratio Test*) *Let $\sum u_k$ be a series with positive terms and suppose that*

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k}$$

(*a*) *If $\rho < 1$, the series converges.*

(*b*) *If $\rho > 1$ or $\rho = +\infty$, the series diverges.*

(*c*) *If $\rho = 1$, the series may converge or diverge, so that another test must be tried.*

Example 25 Each of the following series has positive terms, so the ratio test applies. In each part, use the ratio test to determine whether the following series converge or diverge.

$$\text{(a)} \sum_{k=1}^{\infty} \frac{1}{k!} \qquad \text{(b)} \sum_{k=1}^{\infty} \frac{k}{2^k} \qquad \text{(c)} \sum_{k=1}^{\infty} \frac{k^k}{k!} \qquad \text{(d)} \sum_{k=3}^{\infty} \frac{(2k)!}{4^k} \qquad \text{(e)} \sum_{k=1}^{\infty} \frac{1}{2k-1}$$

Solution (a). The series converges, since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{1/(k+1)!}{1/k!} = \lim_{k\to+\infty} \frac{k!}{(k+1)!} = \lim_{k\to+\infty} \frac{1}{k+1} = 0 < 1$$

Solution (b). The series converges, since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{k+1}{2^{k+1}} \cdot \frac{2^k}{k} = \frac{1}{2} \lim_{k\to+\infty} \frac{k+1}{k} = \frac{1}{2} < 1$$

Solution (c). The series diverges, since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k} = \lim_{k\to+\infty} \frac{(k+1)^k}{k^k} = \lim_{k\to+\infty} \left(1+\frac{1}{k}\right)^k = e > 1$$

See Formula (4) of Section 6.1

$$\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^x = e$$

Solution (d). The series diverges, since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{[2(k+1)]!}{4^{k+1}} \cdot \frac{4^k}{(2k)!} = \lim_{k\to+\infty} \left(\frac{(2k+2)!}{(2k)!} \cdot \frac{1}{4}\right)$$
$$= \lim_{k\to+\infty} \left(\frac{(2k+2)(2k+1)(2k)!}{(2k)!} \cdot \frac{1}{4}\right) = \frac{1}{4} \lim_{k\to+\infty} (2k+2)(2k+1) = +\infty$$

Solution (e). The ratio test is of no help since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{1}{2(k+1)-1} \cdot \frac{2k-1}{1} = \lim_{k\to+\infty} \frac{2k-1}{2k+1} = 1$$

However, the integral test proves that the series diverges since

$$\int_1^{+\infty} \frac{dx}{2x-1} = \lim_{b\to+\infty} \int_1^b \frac{dx}{2x-1} = \lim_{b\to+\infty} \frac{1}{2} \ln(2x-1)\Big]_1^b = +\infty$$

Both the comparison test and the limit comparison test would also have worked here (verify).

3.5.3 The Root Test

THEOREM (*The Root Test*) *Let $\sum u_k$ be a series with positive terms and suppose that*

$$\rho = \lim_{k\to+\infty} \sqrt[k]{u_k} = \lim_{k\to+\infty} (u_k)^{1/k}$$

(*a*) *If $\rho < 1$, the series converges.*

(*b*) *If $\rho > 1$ or $\rho = +\infty$, the series diverges.*

(*c*) *If $\rho = 1$, the series may converge or diverge, so that another test must be tried.*

Example 26 Use the root test to determine whether the following series converge or diverge.

$$\text{(a)} \sum_{k=2}^{\infty} \left(\frac{4k-5}{2k+1}\right)^k \qquad \text{(b)} \sum_{k=1}^{\infty} \frac{1}{(\ln(k+1))^k}$$

Solution (a). The series diverges, since

$$\rho = \lim_{k\to+\infty} (u_k)^{1/k} = \lim_{k\to+\infty} \frac{4k-5}{2k+1} = 2 > 1$$

Solution (b). The series converges, since

$$\rho = \lim_{k\to+\infty} (u_k)^{1/k} = \lim_{k\to+\infty} \frac{1}{\ln(k+1)} = 0 < 1$$

3.5.4 ALTERNATING SERIES; ABSOLUTE AND CONDITIONAL CONVERGENCE

3.5.5 Alternating Series

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k = a_1 - a_2 + a_3 - a_4 + \cdots \tag{1}$$

$$\sum_{k=1}^{\infty}(-1)^{k}a_k = -a_1 + a_2 - a_3 + a_4 - \cdots \tag{2}$$

THEOREM (*Alternating Series Test*) *An alternating series of either form* (1) *or form* (2) *converges if the following two conditions are satisfied:*

(*a*) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_k \geq \cdots$

(*b*) $\lim\limits_{k\to+\infty} a_k = 0$

Example 27 Use the alternating series test to show that the following series converge.

$$\text{(a)} \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} \qquad \text{(b)} \sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+3}{k(k+1)}$$

Solution (a). The two conditions in the alternating series test are satisfied since

$$a_k = \frac{1}{k} > \frac{1}{k+1} = a_{k+1} \quad \text{and} \quad \lim_{k\to+\infty} a_k = \lim_{k\to+\infty}\frac{1}{k} = 0$$

Solution (b). The two conditions in the alternating series test are satisfied since

$$\frac{a_{k+1}}{a_k} = \frac{k+4}{(k+1)(k+2)}\cdot\frac{k(k+1)}{k+3} = \frac{k^2+4k}{k^2+5k+6} = \frac{k^2+4k}{(k^2+4k)+(k+6)} < 1$$

so

$$\boldsymbol{a_k > a_{k+1}}$$

and

$$\lim_{k\to+\infty} a_k = \lim_{k\to+\infty}\frac{k+3}{k(k+1)} = \lim_{k\to+\infty}\frac{\frac{1}{k}+\frac{3}{k^2}}{1+\frac{1}{k}} = 0$$

3.5.6 Absolute Convergence

DEFINITION A series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \cdots + u_k + \cdots$$

is said to ***converge absolutely*** if the series of absolute values

$$\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \cdots + |u_k| + \cdots$$

converges and is said to ***diverge absolutely*** if the series of absolute values diverges.

Example 28 Determine whether the following series converge absolutely.

$$\text{(a)}\ 1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \cdots \qquad \text{(b)}\ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

Solution (a). The series of absolute values is the convergent geometric series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots$$

so the given series converges absolutely.

Solution (b). The series of absolute values is the divergent harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

so the given series diverges absolutely.

THEOREM *If the series*

$$\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \cdots + |u_k| + \cdots$$

converges, then so does the series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \cdots + u_k + \cdots$$

Example 29 Show that the following series converge.

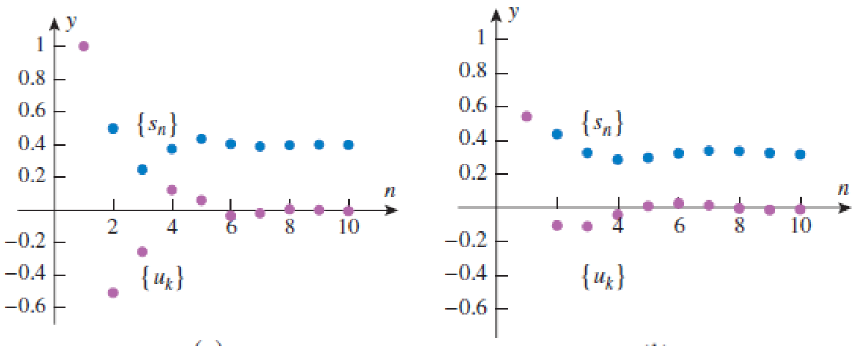
$$\text{(a)}\ 1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \frac{1}{2^6} + \cdots \qquad \text{(b)}\ \sum_{k=1}^{\infty} \frac{\cos k}{k^2}$$

Solution (a). Observe that this is not an alternating series because the signs alternate in pairs after the first term. Thus, we have no convergence test that can be applied directly. However, we showed in Example 28(a) that the series converges absolutely, so Theorem implies that it converges (Figure a).

Solution (b). With the help of a calculating utility, you will be able to verify that the signs of the terms in this series vary irregularly. Thus, we will test for absolute convergence. The series of absolute values is

$$\sum_{k=1}^{\infty}\left|\frac{\cos k}{k^2}\right|$$

$$\left|\frac{\cos k}{k^2}\right| \leq \frac{1}{k^2}$$



But $\Sigma 1/k^2$ is a convergent p-series (p = 2), so the series of absolute values converges by the comparison test. Thus, the given series converges absolutely and hence converges (Figure b).

3.5.7 Conditional Convergence

Example 30 In Example 27(b) we used the alternating series test to show that the series

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+3}{k(k+1)}$$

converges. Determine whether this series converges absolutely or converges conditionally.

Solution. We test the series for absolute convergence by examining the series of absolute values:

$$\sum_{k=1}^{\infty}\left|(-1)^{k+1}\frac{k+3}{k(k+1)}\right| = \sum_{k=1}^{\infty}\frac{k+3}{k(k+1)}$$

Principle 2 suggests that the series of absolute values should behave like the divergent p-series with p = 1. To prove that the series of absolute values diverges, we will apply the limit comparison test with

$$a_k = \frac{k+3}{k(k+1)} \quad \text{and} \quad b_k = \frac{1}{k}$$

We obtain

$$\rho = \lim_{k\to+\infty} \frac{a_k}{b_k} = \lim_{k\to+\infty} \frac{k(k+3)}{k(k+1)} = \lim_{k\to+\infty} \frac{k+3}{k+1} = 1$$

Since ρ is finite and positive, it follows from the limit comparison test that the series of absolute values diverges. Thus, the original series converges and also diverges absolutely, and so converges conditionally.

3.5.8 The Ratio Test for Absolute Convergence

THEOREM (*Ratio Test for Absolute Convergence*) *Let $\sum u_k$ be a series with nonzero terms and suppose that*

$$\rho = \lim_{k\to+\infty} \frac{|u_{k+1}|}{|u_k|}$$

(*a*) *If $\rho < 1$, then the series $\sum u_k$ converges absolutely and therefore converges.*

(*b*) *If $\rho > 1$ or if $\rho = +\infty$, then the series $\sum u_k$ diverges.*

(*c*) *If $\rho = 1$, no conclusion about convergence or absolute convergence can be drawn from this test.*

Example 31 Use the ratio test for absolute convergence to determine whether the series converges.

$$\text{(a)} \sum_{k=1}^{\infty} (-1)^k \frac{2^k}{k!} \qquad \text{(b)} \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!}{3^k}$$

Solution (a). Taking the absolute value of the general term *uk* we obtain

$$|u_k| = \left|(-1)^k \frac{2^k}{k!}\right| = \frac{2^k}{k!}$$

$$\rho = \lim_{k\to+\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to+\infty} \frac{2^{k+1}}{(k+1)!} \cdot \frac{k!}{2^k} = \lim_{k\to+\infty} \frac{2}{k+1} = 0 < 1$$

which implies that the series converges absolutely and therefore converges.

Solution (b). Taking the absolute value of the general term *uk* we obtain

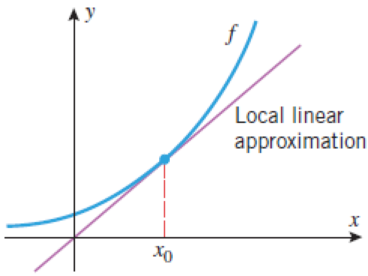
$$|u_k| = \left|(-1)^k \frac{(2k-1)!}{3^k}\right| = \frac{(2k-1)!}{3^k}$$

$$\rho = \lim_{k\to+\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to+\infty} \frac{[2(k+1)-1]!}{3^{k+1}} \cdot \frac{3^k}{(2k-1)!}$$
$$= \lim_{k\to+\infty} \frac{1}{3} \cdot \frac{(2k+1)!}{(2k-1)!} = \frac{1}{3} \lim_{k\to+\infty} (2k)(2k+1) = +\infty$$

which implies that the series diverges.

3.6 MACLAURIN AND TAYLOR POLYNOMIALS

3.6.1 Local Quadratic Approximations



- The local linear approximation of a function f at x_0 is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

- to approximate the function f at x_0 by a polynomial p of **degree 2** with the property that the value of p and the values of its first two derivatives match those of f at x_0.
- The polynomial p is called the ***local quadratic approximation of f at x = x_0***.
- This approximation has the form

$$f(x) \approx c_0 + c_1 x + c_2 x^2$$

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 \quad (1)$$

Example 32 Find the local linear and quadratic approximations of e^x at $x = 0$, and graph ex and the two approximations together.

Solution. If we let $f(x) = e^x$, then $f'(x) = f''(x) = e^x$; and hence

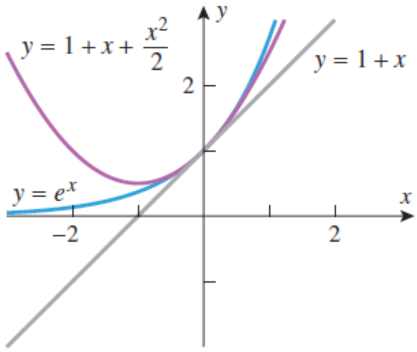
$$f(0) = f'(0) = f''(0) = e^0 = 1$$

Thus, from (1) the local quadratic approximation of e^x at $x = 0$ is

$$e^x \approx 1 + x + \frac{x^2}{2}$$

and the local linear approximation

$$e^x \approx 1 + x$$



The graphs of e^x and the two approximations are shown in Figure. As expected, the local quadratic approximation is more accurate than the local linear approximation near $x = 0$.

3.6.2 Maclaurin Polynomials

It is natural to ask whether one can improve on the accuracy of a local quadratic approximation by using a polynomial of degree 3.

Problem Given a function f that can be differentiated n times at $x = x_0$, find a polynomial p of degree n with the property that the value of p and the values of its first n derivatives match those of f at x_0.

We will begin by solving this problem in the case where $x_0 = 0$. Thus, we want a polynomial

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n$$

Definition If f can be differentiated n times at 0, then we define the ***nth Maclaurin polynomial for f*** to be

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

Example 33 Find the Maclaurin polynomials p_0, p_1, p_2, p_3, and p_n for e^x .

Solution. Let $f(x) = e^x$. Thus,

$$f'(x) = f''(x) = f'''(x) = \cdots = f^{(n)}(x) = e^x$$

$$f(0) = f'(0) = f''(0) = f'''(0) = \cdots = f^{(n)}(0) = e^0 = 1$$

$$\begin{aligned}
p_0(x) &= f(0) = 1 \\
p_1(x) &= f(0) + f'(0)x = 1 + x \\
p_2(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{x^2}{2!} = 1 + x + \frac{1}{2}x^2 \\
p_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \\
p_n(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\
&= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}
\end{aligned}$$

Example 34 Find the nth Maclaurin polynomials for

(a) $\sin x$ (b) $\cos x$

Solution (a). In the Maclaurin polynomials for $\sin x$, only the odd powers of x appear explicitly. To see this, let $f(x) = \sin x$; thus,

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \end{aligned}$$

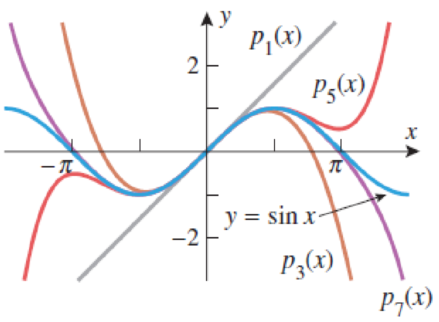
Since $f^{(4)}(x) = \sin x = f(x)$, the pattern 0, 1, 0, −1 will repeat as we evaluate successive derivatives at 0. Therefore, the successive Maclaurin polynomials for $\sin x$ are

$$\begin{aligned} p_0(x) &= 0 \\ p_1(x) &= 0 + x \\ p_2(x) &= 0 + x + 0 \\ p_3(x) &= 0 + x + 0 - \frac{x^3}{3!} \\ p_4(x) &= 0 + x + 0 - \frac{x^3}{3!} + 0 \\ p_5(x) &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} \\ p_6(x) &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 \\ p_7(x) &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} \end{aligned}$$

Because of the zero terms, each even-order Maclaurin polynomial [after $p_0(x)$] is the same as the preceding odd-order Maclaurin polynomial. That is,

$$p_{2k+1}(x) = p_{2k+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad (k = 0, 1, 2, \ldots)$$

The graphs of $\sin x$, $p_1(x)$, $p_3(x)$, $p_5(x)$, and $p_7(x)$ are shown in Figure.



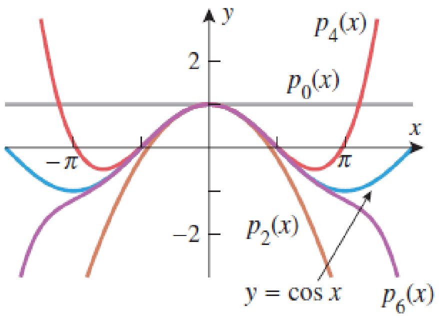
Solution (b). In the Maclaurin polynomials for cos x, only the even powers of x appear explicitly; the computations are similar to that in part (a). The reader should be able to show that

$$p_0(x) = p_1(x) = 1$$
$$p_2(x) = p_3(x) = 1 - \frac{x^2}{2!}$$
$$p_4(x) = p_5(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$
$$p_6(x) = p_7(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

In general, the Maclaurin polynomials for cos x are given by

$$p_{2k}(x) = p_{2k+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} \quad (k = 0, 1, 2, \ldots)$$

The graphs of cos x, $p_0(x)$, $p_2(x)$, $p_4(x)$, and $p_6(x)$ are shown in Figure.



3.6.3 Taylor Polynomials

- Up to now we have focused on approximating a function f in the vicinity of $x = 0$.
- Now we will consider the more general case of approximating f in the vicinity of an arbitrary domain value x_0.

Definition If f can be differentiated n times at x_0, then we define the ***nth Taylor polynomial for f about x = x₀*** to be

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

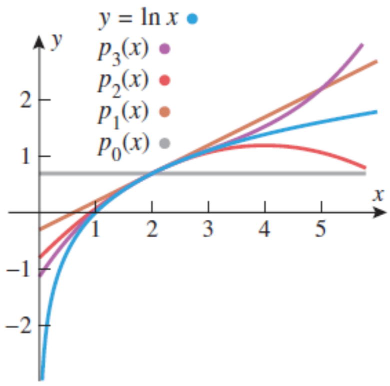
Example 35 Find the first four Taylor polynomials for ln *x* about *x* = 2.

Solution. Let *f(x)* = ln *x*. Thus,

$$\begin{aligned} f(x) &= \ln x & f(2) &= \ln 2 \\ f'(x) &= 1/x & f'(2) &= 1/2 \\ f''(x) &= -1/x^2 & f''(2) &= -1/4 \\ f'''(x) &= 2/x^3 & f'''(2) &= 1/4 \end{aligned}$$

$$\begin{aligned} p_0(x) &= f(2) = \ln 2 \\ p_1(x) &= f(2) + f'(2)(x-2) = \ln 2 + \tfrac{1}{2}(x-2) \\ p_2(x) &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 = \ln 2 + \tfrac{1}{2}(x-2) - \tfrac{1}{8}(x-2)^2 \\ p_3(x) &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 \\ &= \ln 2 + \tfrac{1}{2}(x-2) - \tfrac{1}{8}(x-2)^2 + \tfrac{1}{24}(x-2)^3 \end{aligned}$$

The graph of ln *x* (in blue) and its first four Taylor polynomials about *x* = 2 are shown in Figure. As expected, these polynomials produce their best approximations of ln *x* near 2.



3.7 MACLAURIN AND TAYLOR SERIES; POWER SERIES

3.7.1 Maclaurin and Taylor Series

DEFINITION If f has derivatives of all orders at x_0, then we call the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \cdots \quad (1)$$

the ***Taylor series for f about $x = x_0$***. In the special case where $x_0 = 0$, this series becomes

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots \quad (2)$$

in which case we call it the ***Maclaurin series for f***.

Example 36 Find the Maclaurin series for

(a) e^x (b) $\sin x$ (c) $\cos x$

Solution (a). From previous example, we found that the nth Maclaurin polynomial for e^x is

$$p_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

Thus, the Maclaurin series for e^x is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots$$

Solution (b). From previous example, we found that the Maclaurin polynomials for $\sin x$ are given by

$$p_{2k+1}(x) = p_{2k+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad (k = 0, 1, 2, \ldots)$$

Thus, the Maclaurin series for $\sin x$ is

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \cdots$$

Solution (c). From previous example, we found that the Maclaurin polynomials for $\cos x$ are given by

$$p_{2k}(x) = p_{2k+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} \quad (k = 0, 1, 2, \ldots)$$

Thus, the Maclaurin series for $\cos x$ is

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots$$

3.7.2 Power Series in x

If $c_0, c_1, c_2, \ldots$ are constants and x is a variable, then a series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k + \cdots$$

is called a ***power series in x***. Some examples are

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

These are the Maclaurin series for the functions $1/(1-x)$, e^x, and $\cos x$, respectively. Indeed, every Maclaurin series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(k)}(0)}{k!} x^k + \cdots$$

is a power series in x.

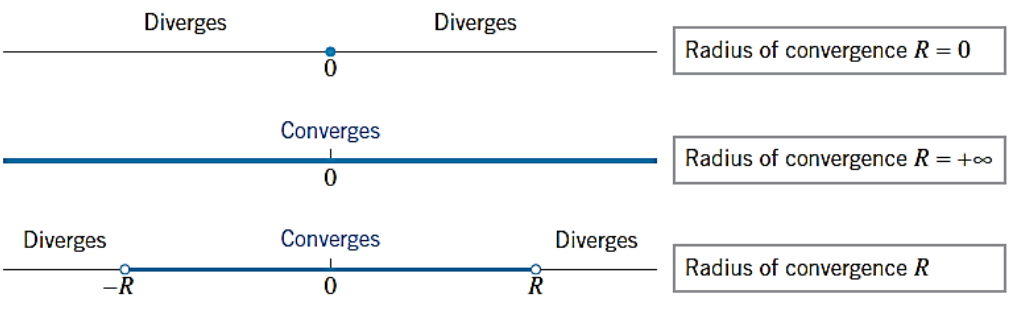
3.7.3 Radius and Interval of Convergence

THEOREM *For any power series in x, exactly one of the following is true:*

(a) *The series converges only for $x = 0$.*

(b) *The series converges absolutely (and hence converges) for all real values of x.*

(c) *The series converges absolutely (and hence converges) for all x in some finite open interval $(-R, R)$ and diverges if $x < -R$ or $x > R$. At either of the values $x = R$ or $x = -R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.*



3.7.4 Finding the Interval of Convergence

Example 37 Find the interval of convergence and radius of convergence of the following power series.

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \sum_{k=0}^{\infty} k!x^k$$

Solution (1). Applying the ratio test for absolute convergence to the given series, we obtain

$$\rho = \lim_{k\to+\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to+\infty}\left|\frac{x^{k+1}}{(k+1)!}\cdot\frac{k!}{x^k}\right| = \lim_{k\to+\infty}\left|\frac{x}{k+1}\right| = 0$$

Since $\rho < 1$ for all x, the series converges absolutely for all x. Thus, the interval of convergence is $(-\infty, +\infty)$ and the radius of convergence is $R = +\infty$.

Solution (2). If $x \neq 0$, then the ratio test for absolute convergence yields

$$\rho = \lim_{k\to+\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to+\infty}\left|\frac{(k+1)!x^{k+1}}{k!x^k}\right| = \lim_{k\to+\infty}|(k+1)x| = +\infty$$

Therefore, the series diverges for all nonzero values of x. Thus, the interval of convergence is the single value $x = 0$ and the radius of convergence is $R = 0$.

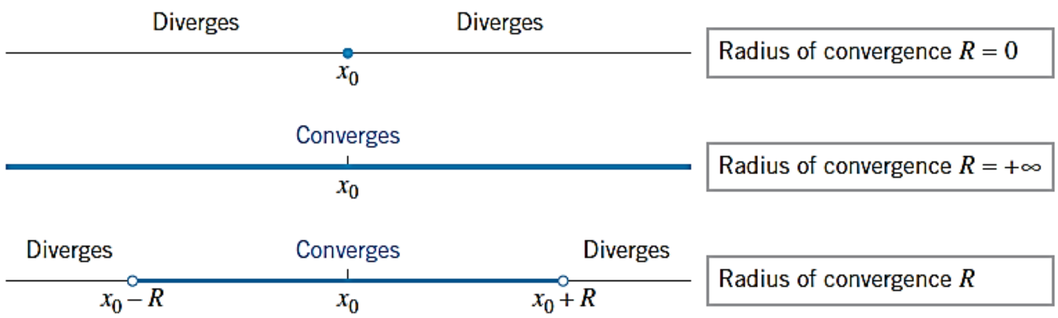
3.7.5 Power Series in $x - x_0$

THEOREM *For a power series $\sum c_k(x-x_0)^k$, exactly one of the following statements is true:*

(*a*) *The series converges only for $x = x_0$.*

(*b*) *The series converges absolutely (and hence converges) for all real values of x.*

(*c*) *The series converges absolutely (and hence converges) for all x in some finite open interval $(x_0 - R, x_0 + R)$ and diverges if $x < x_0 - R$ or $x > x_0 + R$. At either of the values $x = x_0 - R$ or $x = x_0 + R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.*



Example 38 Find the interval of convergence and radius of convergence of the series

$$\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2}$$

Solution. We apply the ratio test for absolute convergence.

$$\begin{aligned}
\rho = \lim_{k \to +\infty} \left| \frac{u_{k+1}}{u_k} \right| &= \lim_{k \to +\infty} \left| \frac{(x-5)^{k+1}}{(k+1)^2} \cdot \frac{k^2}{(x-5)^k} \right| \\
&= \lim_{k \to +\infty} \left[|x-5| \left(\frac{k}{k+1} \right)^2 \right] \\
&= |x-5| \lim_{k \to +\infty} \left(\frac{1}{1+(1/k)} \right)^2 = |x-5|
\end{aligned}$$

Thus, the series converges absolutely if $|x-5| < 1$, or $-1 < x-5 < 1$, or $4 < x < 6$. The series diverges if $x < 4$ or $x > 6$.

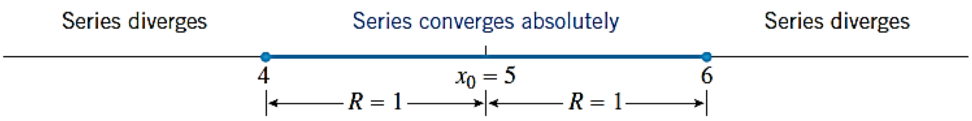
To determine the convergence behaviour at the endpoints $x = 4$ and $x = 6$, we substitute these values in the given series. If $x = 6$, the series becomes

$$\sum_{k=1}^{\infty} \frac{1^k}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

which is a convergent p-series ($p = 2$). If $x = 4$, the series becomes

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots$$

Since this series converges absolutely, the interval of convergence for the given series is [4, 6]. The radius of convergence is $R = 1$ (Figure).



3.7.6 Estimating the *n*th Remainder

Rn(x) denote the difference between *f(x)* and its *n*th Taylor polynomial; that is,

$$R_n(x) = f(x) - p_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

THEOREM (*The Remainder Estimation Theorem*) *If the function f can be differentiated $n+1$ times on an interval containing the number x_0, and if M is an upper bound for $|f^{(n+1)}(x)|$ on the interval, that is, $|f^{(n+1)}(x)| \le M$ for all x in the interval, then*

$$|R_n(x)| \le \frac{M}{(n+1)!}|x - x_0|^{n+1}$$

for all x in the interval.

THEOREM *The equality*

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

holds at a point x if and only if $\lim_{n\to+\infty} R_n(x) = 0$.

Example 39 Use an *n*th Maclaurin polynomial for e^x to approximate *e* to five decimal place accuracy.

Solution. We note first that the exponential function e^x has derivatives of all orders for every real number *x*. From previous example, the *n*th Maclaurin polynomial for e^x is

$$\sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

from which we have

$$e = e^1 \approx \sum_{k=0}^{n} \frac{1^k}{k!} = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

Thus, our problem is to determine how many terms to include in a Maclaurin polynomial for e^x to achieve five decimal-place accuracy; that is, we want to choose *n* so that the absolute value of the *n*th remainder at $x = 1$ satisfies

$$|R_n(1)| \le 0.000005$$

To determine *n* we use the Remainder Estimation Theorem with $f(x) = e^x$, $x = 1$, $x_0 = 0$, and the interval [0, 1]. In this case it follows that

$$|R_n(1)| \le \frac{M}{(n+1)!} \cdot |1 - 0|^{n+1} = \frac{M}{(n+1)!}$$

where M is an upper bound on the value of $f^{(n+1)}(x) = e^x$ for x in the interval [0, 1]. However, e^x is an increasing function, so its maximum value on the interval [0, 1] occurs at $x = 1$; that is, $e^x \le e$ on this interval. Thus, we can take $M = e$ to obtain

$$|R_n(1)| \le \frac{e}{(n+1)!}$$

Unfortunately, this inequality is not very useful because it involves e, which is the very quantity we are trying to approximate. However, if we accept that $e < 3$, then we can replace (previous equation) with the following less precise, but more easily applied, inequality:

$$|R_n(1)| \le \frac{3}{(n+1)!}$$

Thus, we can achieve five decimal-place accuracy by choosing n so that

$$\frac{3}{(n+1)!} \le 0.000005 \quad \text{or} \quad (n+1)! \ge 600{,}000$$

Since 9! = 362,880 and 10! = 3,628,800, the smallest value of n that meets this criterion is n = 9. Thus, to five decimal-place accuracy

$$e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} \approx 2.71828$$

As a check, a calculator's 12-digit representation of e is $e \approx 2.71828182846$, which agrees with the preceding approximation when rounded to five decimal places.

3.8 DIFFERENTIATING AND INTEGRATING POWER SERIES; MODELING WITH TAYLOR SERIES

3.8.1 Differentiating Power Series

THEOREM (*Differentiation of Power Series*) *Suppose that a function f is represented by a power series in $x - x_0$ that has a nonzero radius of convergence R; that is,*

$$f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \qquad (x_0 - R < x < x_0 + R)$$

Then:

(*a*) *The function f is differentiable on the interval $(x_0 - R, x_0 + R)$.*

(*b*) *If the power series representation for f is differentiated term by term, then the resulting series has radius of convergence R and converges to f' on the interval $(x_0 - R, x_0 + R)$; that is,*

$$f'(x) = \sum_{k=0}^{\infty} \frac{d}{dx}[c_k (x - x_0)^k] \qquad (x_0 - R < x < x_0 + R)$$

THEOREM *If a function f can be represented by a power series in $x - x_0$ with a nonzero radius of convergence R, then f has derivatives of all orders on the interval $(x_0 - R, x_0 + R)$.*

Example 40 we showed that the 4 function $J_0(x)$, represented by the power series

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2} \qquad (1)$$

has radius of convergence $+\infty$. Thus, $J_0(x)$ has derivatives of all orders on the interval $(-\infty, +\infty)$, and these can be obtained by differentiating the series term by term. For example, if we write (1) as

$$J_0(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}$$

and differentiate term by term, we obtain

$$J_0'(x) = \sum_{k=1}^{\infty} \frac{(-1)^k (2k) x^{2k-1}}{2^{2k}(k!)^2} = \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k-1}}{2^{2k-1} k!(k-1)!}$$

3.8.2 Integrating Power Series

THEOREM (*Integration of Power Series*) *Suppose that a function f is represented by a power series in $x - x_0$ that has a nonzero radius of convergence R; that is,*

$$f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \qquad (x_0 - R < x < x_0 + R)$$

(*a*) *If the power series representation of f is integrated term by term, then the resulting series has radius of convergence R and converges to an antiderivative for $f(x)$ on the interval $(x_0 - R, x_0 + R)$; that is,*

$$\int f(x)\,dx = \sum_{k=0}^{\infty} \left[\frac{c_k}{k+1}(x - x_0)^{k+1}\right] + C \qquad (x_0 - R < x < x_0 + R)$$

(*b*) *If α and β are points in the interval $(x_0 - R, x_0 + R)$, and if the power series representation of f is integrated term by term from α to β, then the resulting series converges absolutely on the interval $(x_0 - R, x_0 + R)$ and*

$$\int_{\alpha}^{\beta} f(x)\,dx = \sum_{k=0}^{\infty} \left[\int_{\alpha}^{\beta} c_k (x - x_0)^k\,dx\right]$$

3.8.3 Some Practical Ways to Find Taylor Series

Example 41 Find Taylor series for the given functions about the given x_0.

$$\text{(a) } e^{-x^2}, \quad x_0 = 0 \qquad \text{(b) } \ln x, \quad x_0 = 1$$

Solution (a). The simplest way to find the Maclaurin series for e^{-x^2} is to substitute $-x^2$ for x in the Maclaurin Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

to obtain

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$$

Since converges for all values of x, so will the series for e^{-x^2}.

Solution (b). We begin with the Maclaurin series for $\ln(1 + x)$,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad (-1 < x \le 1)$$

Substituting $x - 1$ for x in this series gives

$$\ln(1 + [x-1]) = \ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

Since the original series converges when $-1 < x \leq 1$, the interval of convergence will be $-1 < x - 1 \leq 1$ or, equivalently, $0 < x \leq 2$.

Example 42 Find the Maclaurin series for $\tan^{-1} x$.

Solution. It would be tedious to find the Maclaurin series directly. A better approach is to start with the formula

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

and integrate the Maclaurin series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots \qquad (-1 < x < 1)$$

term by term. This yields

$$\tan^{-1} x + C = \int \frac{1}{1+x^2}\,dx = \int [1 - x^2 + x^4 - x^6 + x^8 - \cdots]\,dx$$

or

$$\tan^{-1} x = \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots\right] - C$$

The constant of integration can be evaluated by substituting $x = 0$ and using the condition $\tan^{-1} 0 = 0$. This gives $C = 0$, so that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \qquad (-1 < x < 1)$$

3.8.4 Approximating Definite Integrals Using Taylor Series

Example 43 Approximate the integral

$$\int_0^1 e^{-x^2}\,dx$$

To three decimal-place accuracy by expanding the integrand in a Maclaurin series and integrating term by term.

Solution. We found in Example 41(a) that the Maclaurin series for e^{-x^2} is

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$$

Therefore,

$$\begin{aligned}\int_0^1 e^{-x^2}\,dx &= \int_0^1 \left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots\right] dx \\ &= \left[x - \frac{x^3}{3} + \frac{x^5}{5(2!)} - \frac{x^7}{7(3!)} + \frac{x^9}{9(4!)} - \cdots\right]_0^1 \\ &= 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} - \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!}\end{aligned}$$

-

Since this series clearly satisfies the hypotheses of the alternating series test, it follows from Theorem that if we approximate the integral by s_n (the nth partial sum of the series), then

$$\left|\int_0^1 e^{-x^2}\,dx - s_n\right| < \frac{1}{[2(n+1)+1](n+1)!} = \frac{1}{(2n+3)(n+1)!}$$

Thus, for three decimal-place accuracy we must choose n such that

$$\frac{1}{(2n+3)(n+1)!} \le 0.0005 = 5 \times 10^{-4}$$

With the help of a calculating utility you can show that the smallest value of n that satisfies this condition is $n = 5$. Thus, the value of the integral to three decimal-place accuracy is

$$\int_0^1 e^{-x^2}\,dx \approx 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} - \frac{1}{11\cdot 5!} \approx 0.747$$

3.8.5 Finding Taylor Series by Multiplication and Division

Example 44 Find the first three nonzero terms in the Maclaurin series for the function

$$f(x) = e^{-x^2}\tan^{-1} x.$$

Solution. Using the series for e^{-x2} and $\tan^{-1} x$ obtained in previous examples gives

$$e^{-x^2}\tan^{-1} x = \left(1 - x^2 + \frac{x^4}{2} - \cdots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right)$$

Multiplying, as shown in the margin, we obtain

$$e^{-x^2}\tan^{-1} x = x - \frac{4}{3}x^3 + \frac{31}{30}x^5 - \cdots$$

$$
\begin{array}{rl}
 & 1 - x^2 + \dfrac{x^4}{2} - \cdots \\
\times & x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots \\
\hline
 & x - x^3 + \dfrac{x^5}{2} - \cdots \\
 & \quad - \dfrac{x^3}{3} + \dfrac{x^5}{3} - \dfrac{x^7}{6} + \cdots \\
 & \qquad\qquad \dfrac{x^5}{5} - \dfrac{x^7}{5} + \cdots \\
\hline
 & x - \frac{4}{3}x^3 + \frac{31}{30}x^5 - \cdots
\end{array}
$$

Example 45 Find the first three nonzero terms in the Maclaurin series for tan *x*.

Solution. Using the first three terms in the Maclaurin series for sin *x* and cos *x*, we can express tan *x* as

$$
\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots}
$$

Dividing, as shown in the margin, we obtain

$$
\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots
$$

$$
\begin{array}{r|l}
 & x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \cdots \\
\cline{2-2}
1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \cdots & x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \cdots \\
 & x - \dfrac{x^3}{2} + \dfrac{x^5}{24} - \cdots \\
\cline{2-2}
 & \qquad \dfrac{x^3}{3} - \dfrac{x^5}{30} + \cdots \\
 & \qquad \dfrac{x^3}{3} - \dfrac{x^5}{6} + \cdots \\
\cline{2-2}
 & \qquad\qquad \dfrac{2x^5}{15} + \cdots
\end{array}
$$

University of Anbar

College of Engineering

Civil Engineering Department

LECTURE NOTE

COURSE CODE- CE 1202

CALCULUS II

Chapter Four

By

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CHAPTER FOUR

PARAMETRIC AND POLAR CURVES

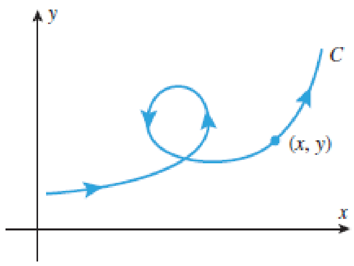
4.1 PARAMETRIC EQUATIONS; TANGENT LINES AND ARC LENGTH FOR PARAMETRIC CURVES

4.1.1 Parametric Equations

Suppose that a particle moves along a curve C in the xy-plane in such a way that its x- and y-coordinates, as functions of time, are

$$x = f(t), \qquad y = g(t)$$

We call these the ***parametric equations*** of motion for the particle and refer to C as the ***trajectory*** of the particle or the ***graph*** of the equations (Figure 4-1). The variable t is called the ***parameter*** for the equations.

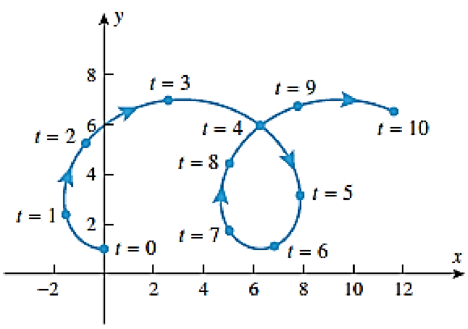


A moving particle with trajectory C

Example 1 Sketch the trajectory over the time interval $0 \le t \le 10$ of the particle whose parametric equations of motion are

$$x = t - 3\sin t, \qquad y = 4 - 3\cos t$$

Solution.



t	x	y
0	0.0	1.0
1	−1.5	2.4
2	−0.7	5.2
3	2.6	7.0
4	6.3	6.0
5	7.9	3.1
6	6.8	1.1
7	5.0	1.7
8	5.0	4.4
9	7.8	6.7
10	11.6	6.5

Figure 4-1

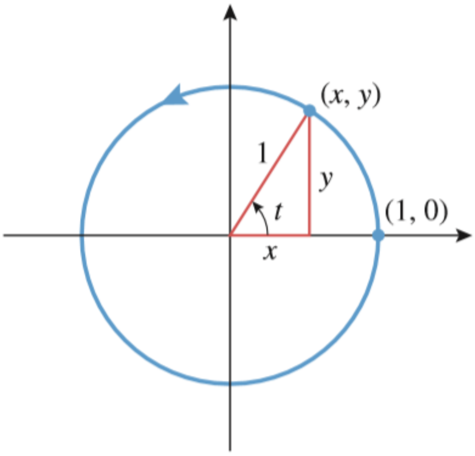
Example 2 : Find the graph of the parametric equations

$$x = \cos t, \qquad y = \sin t \qquad (0 \le t \le 2\pi)$$

Solution.

One way to find the graph is to eliminate the parameter t by noting that

$$x^2 + y^2 = \sin^2 t + \cos^2 t = 1$$



$x = \cos t,\ y = \sin t$
$(0 \le t \le 2\pi)$

Thus, the graph is contained in the unit circle $x^2 + y^2 = 1$. Geometrically, the parameter t can be interpreted as the angle swept out by the radial line from the origin to the point $(x, y) = (\cos t, \sin t)$ on the unit circle (Figure). As t increases from 0 to 2π, the point traces the circle counterclockwise, starting at (1, 0) when $t = 0$ and completing one full revolution when $t = 2\pi$. One can obtain different portions of the circle by varying the interval over which the parameter varies.

4.1.2 Tangent Lines To Parametric Curves

We will be concerned with curves that are given by parametric equations

$$x = f(t), \qquad y = g(t)$$

in which $f(t)$ and $g(t)$ have continuous first derivatives with respect to t. It can be proved that if $dx/dt \neq 0$, then y is a differentiable function of x, in which case the chain rule implies that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

This formula makes it possible to find dy/dx directly from the parametric equations without eliminating the parameter.

Example 3

Find the slope of the tangent line to the unit circle

$$x = \cos t,\ y = \sin t \quad (0 \le t \le 2\pi)$$

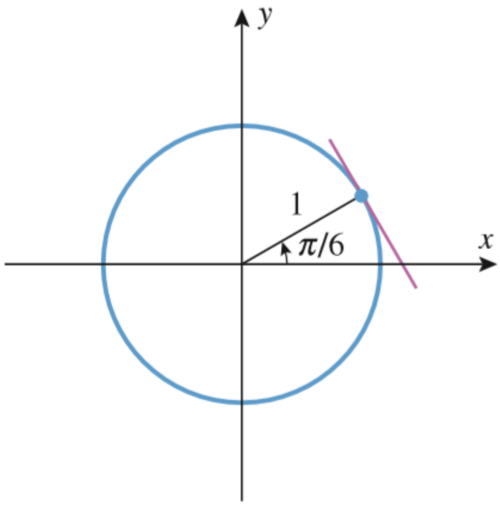
at the point where $t = \pi/6$ (Figure).

Solution. the slope at a general point on the circle is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t} = -\cot t$$

Thus, the slope at $t = \pi/6$ is

$$\left.\frac{dy}{dx}\right|_{t=\pi/6} = -\cot\frac{\pi}{6} = -\sqrt{3}$$



Example 4

In a disastrous first flight, an experimental paper airplane follows the trajectory of the particle in Example 1:

$$x = t - 3\sin t, \qquad y = 4 - 3\cos t \qquad (t \ge 0)$$

but crashes into a wall at time $t = 10$ (Figure).

(a) At what times was the airplane flying horizontally?

(b) At what times was it flying vertically?

Solution (a). The airplane was flying horizontally at those times when $dy/dt = 0$ and $dx/dt \neq 0$. From the given trajectory we have

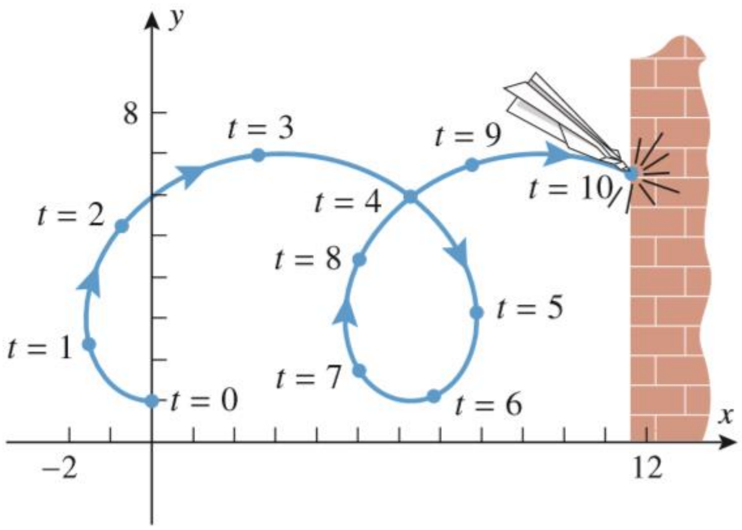
$$\frac{dy}{dt} = 3\sin t \quad \text{and} \quad \frac{dx}{dt} = 1 - 3\cos t$$

Setting $dy/dt = 0$ yields the equation $3\sin t = 0$, or, more simply, $\sin t = 0$. This equation has four solutions in the time interval $0 \le t \le 10$:

$$t = 0, \quad t = \pi, \quad t = 2\pi, \quad t = 3\pi$$

Since $dx/dt = 1 - 3\cos t \ne 0$ for these values of t (verify), the airplane was flying horizontally at times

$$t = 0, \quad t = \pi \approx 3.14, \quad t = 2\pi \approx 6.28, \quad \text{and} \quad t = 3\pi \approx 9.42$$

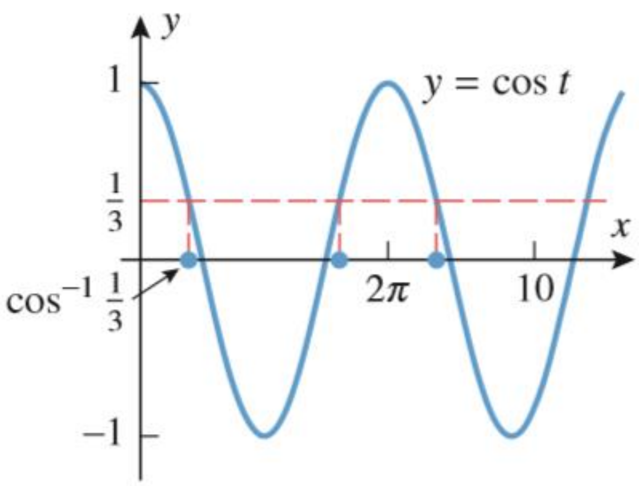


Solution (b). The airplane was flying vertically at those times when $dx/dt = 0$ and $dy/dt \ne 0$.

$$1 - 3\cos t = 0 \quad \text{or} \quad \cos t = \tfrac{1}{3}$$

This equation has three solutions in the time interval $0 \le t \le 10$.

$$t = \cos^{-1}\tfrac{1}{3}, \quad t = 2\pi - \cos^{-1}\tfrac{1}{3}, \quad t = 2\pi + \cos^{-1}\tfrac{1}{3}$$



Although it is true that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

you cannot conclude that d^2y/dx^2 is the quotient of d^2y/dt^2 and d^2x/dt^2. To illustrate that this conclusion is erroneous, show that for the parametric curve in Example 7,

$$\left.\frac{d^2y}{dx^2}\right|_{t=1} \neq \left.\frac{d^2y/dt^2}{d^2x/dt^2}\right|_{t=1}$$

Example 7

Without eliminating the parameter, find dy/dx and d^2y/dx^2 at (1, 1) and (1, −1) on the semicubical parabola given by the parametric equations. $x = t^2$, $y = t^3$

Solution.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3}{2}t \qquad (t \neq 0)$$

applied to $y' = dy/dx$ we have

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{3/2}{2t} = \frac{3}{4t}$$

Since the point (1, 1) on the curve corresponds to $t = 1$ in the parametric equations, it

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{3}{2} \quad \text{and} \quad \left.\frac{d^2y}{dx^2}\right|_{t=1} = \frac{3}{4}$$

Similarly, the point (1, −1) corresponds to $t = -1$ in the parametric equations, so applying

$$\left.\frac{dy}{dx}\right|_{t=-1} = -\frac{3}{2} \quad \text{and} \quad \left.\frac{d^2y}{dx^2}\right|_{t=-1} = -\frac{3}{4}$$

4.1.3 Arc Length of Parametric Curves

ARC LENGTH FORMULA FOR PARAMETRIC CURVES If no segment of the curve represented by the parametric equations

$$x = x(t), \quad y = y(t) \qquad (a \leq t \leq b)$$

is traced more than once as t increases from a to b, and if dx/dt and dy/dt are continuous functions for $a \leq t \leq b$, then the arc length L of the curve is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 8

Find the circumference of a circle of radius ***a*** from the parametric equations

$$x = a\cos t, \qquad y = a\sin t \qquad (0 \le t \le 2\pi)$$

Solution.

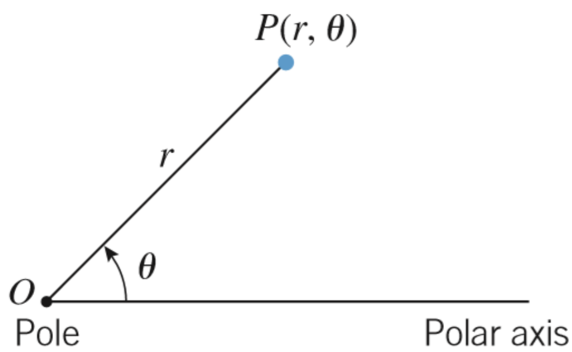
$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (a\cos t)^2}\,dt$$
$$= \int_0^{2\pi} a\,dt = at\Big]_0^{2\pi} = 2\pi a$$

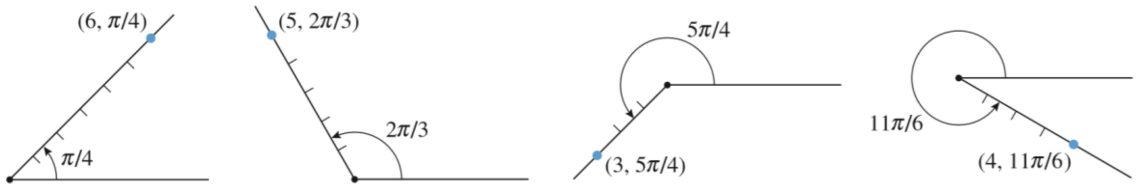
4.2 POLAR COORDINATES

4.2.1 Polar Coordinate Systems

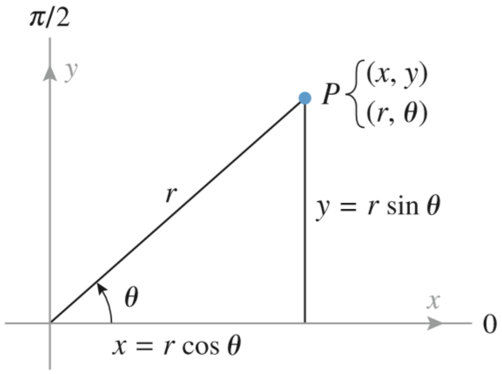
A ***polar coordinate system*** in a plane consists of a fixed point O, called the ***pole*** (or ***origin***), and a ray emanating from the pole, called the ***polar axis***.

we can associate with each point P in the plane a pair of polar coordinates (r, θ), where r is the distance from P to the pole and θ is an angle from the polar axis to the ray OP (Figure). The number r is called the radial coordinate of P and the number θ the angular coordinate(or polar angle) of P.





As suggested by Figure, these coordinates are related by the equations



$$x = r\cos\theta, \qquad y = r\sin\theta \tag{1}$$

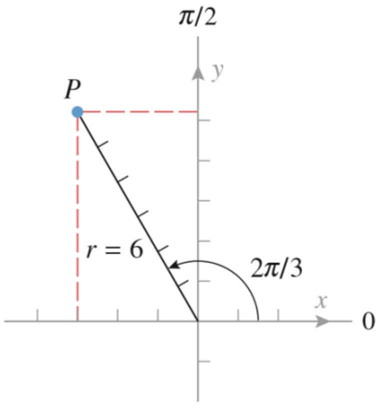
These equations are well suited for finding x and y when r and θ are known. However, to find r and θ when x and y are known, it is preferable to use the identities $\sin^2\theta + \cos^2\theta = 1$ and $\tan\theta = \sin\theta / \cos\theta$ to rewrite (1) as

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x} \tag{2}$$

Example 9

Find the rectangular coordinates of the point P whose polar coordinates are $(r, \theta) = (6, 2\pi/3)$ (Figure).

Solution. Substituting the polar coordinates $r = 6$ and $\theta = 2\pi/3$ in (1) yields



$$x = 6\cos\frac{2\pi}{3} = 6\left(-\frac{1}{2}\right) = -3$$

$$y = 6\sin\frac{2\pi}{3} = 6\left(\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$$

Thus, the rectangular coordinates of P are $(x, y) = (-3, 3\sqrt{3})$.

Example 10

Find polar coordinates of the point P whose rectangular coordinates are $(-2, -2\sqrt{3})$ (Figure).

Solution.

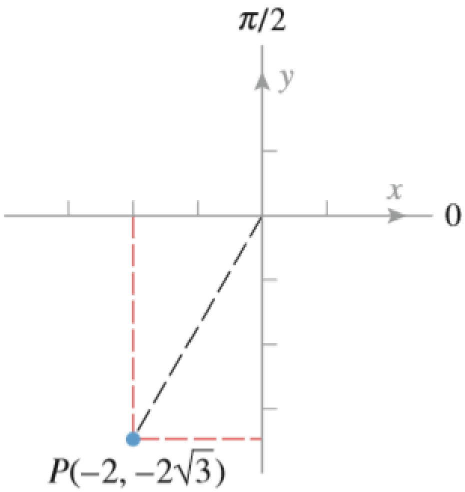
We will find the polar coordinates (r, θ) of P that satisfy the conditions $r > 0$ and $0 \le \theta < 2\pi$.

$$r^2 = x^2 + y^2 = (-2)^2 + (-2\sqrt{3})^2 = 4 + 12 = 16$$

so $r = 4$. From the second equation in (2),

$$\tan\theta = \frac{y}{x} = \frac{-2\sqrt{3}}{-2} = \sqrt{3}$$

From this and the fact that $(-2, -2\sqrt{3})$ lies in the third quadrant, it follows that the angle satisfying the requirement $0 \le \theta < 2\pi$ is $\theta = 4\pi/3$. Thus, $(r, \theta) = (4, 4\pi/3)$ are polar coordinates of P. All other polar coordinates of P are expressible in the form



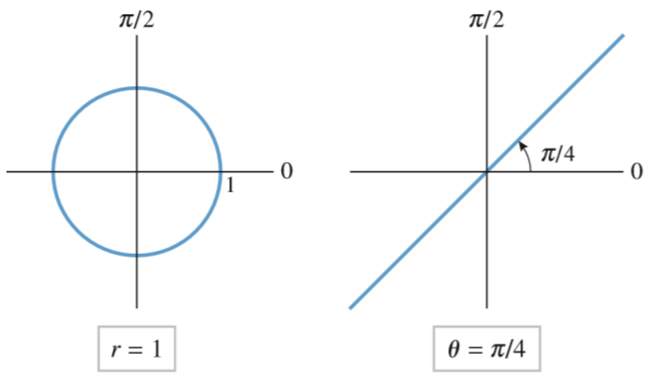
4.2.2 Graphs in Polar Coordinates

Example 11

Sketch the graphs of (a) $r = 1$ (b) $\theta = \pi/4$ in polar coordinates.

Solution (a). For all values of θ, the point $(1, \theta)$is 1 unit away from the pole. Since θ is arbitrary, the graph is the circle of radius 1 centered at the pole (Figure a).

Solution (b). For all values of r, the point $(r, \pi/4)$ lies on a line that makes an angle of $\pi/4$ with the polar axis (Figure b). Positive values of r correspond to points on the line in the first quadrant and negative values of r to points on the line in the third quadrant. Thus, in absence of any restriction on r, the graph is the entire line. Observe, however, that had we imposed the restriction $r \ge 0$, the graph would have been just the ray in the first quadrant.



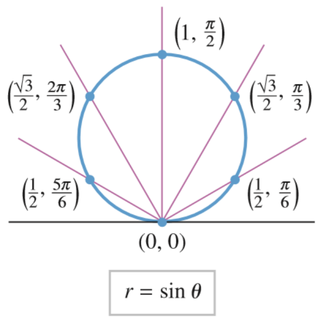
(a) (b)

Example 12

Sketch the graph of the equation $r = \sin\theta$ in polar coordinates by plotting points

Solution. Table 1 shows the coordinates of points on the graph at increments of π/6. These points are plotted in Figure1. Note, however, that there are 13 points listed in the table but only 6 distinct plotted points. This is because the pairs from $\theta = \pi$ on yield

θ (RADIANS)	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$r = \sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0
(r, θ)	$(0, 0)$	$\left(\frac{1}{2}, \frac{\pi}{6}\right)$	$\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$	$\left(1, \frac{\pi}{2}\right)$	$\left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$	$\left(\frac{1}{2}, \frac{5\pi}{6}\right)$	$(0, \pi)$	$\left(-\frac{1}{2}, \frac{7\pi}{6}\right)$	$\left(-\frac{\sqrt{3}}{2}, \frac{4\pi}{3}\right)$	$\left(-1, \frac{3\pi}{2}\right)$	$\left(-\frac{\sqrt{3}}{2}, \frac{5\pi}{3}\right)$	$\left(-\frac{1}{2}, \frac{11\pi}{6}\right)$	$(0, 2\pi)$



Observe that the points in Figure appear to lie on a circle. We can confirm that this is so by expressing the polar equation $r = \sin\theta$ in terms of x and y. To do this, we multiply the equation through by r to obtain

$$r^2 = r \sin\theta$$

which now allows us to apply Formulas (1) and (2) to rewrite the equation as

$$x^2 + y^2 = y$$

Rewriting this equation as $x^2 + y^2 - y = 0$ and then completing the square yields

$$x^2 + \left(y - \tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

which is a circle of radius $\frac{1}{2}$ centered at the point $\left(0, \frac{1}{2}\right)$ in the xy-plane.

Example 13

Sketch the graph of $r = \cos 2\theta$ in polar coordinates

Solution. Instead of plotting points, we will use the graph of $r = \cos 2\theta$ in rectangular coordinates (Figure a) to visualize how the polar graph of this equation is generated. The analysis and the resulting polar graph are shown in Figure b. This curve is called a ***four-petal rose***.

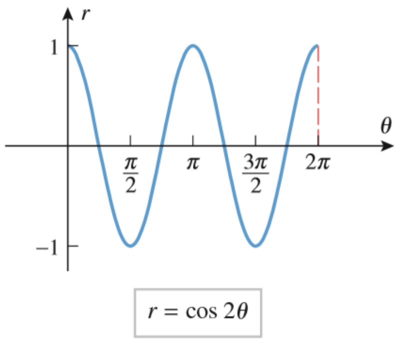


Figure a

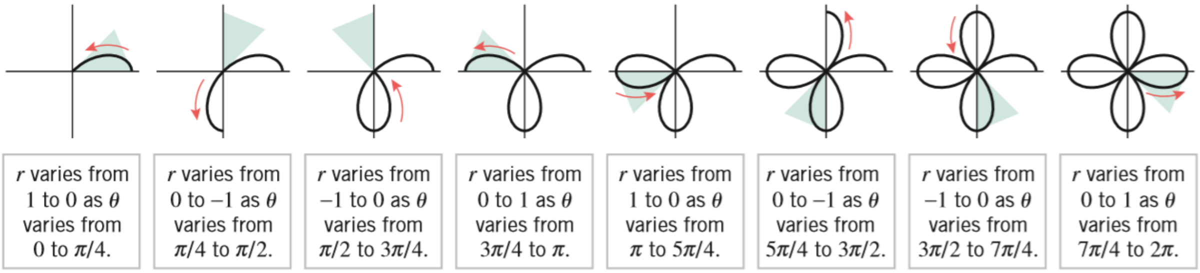


Figure b

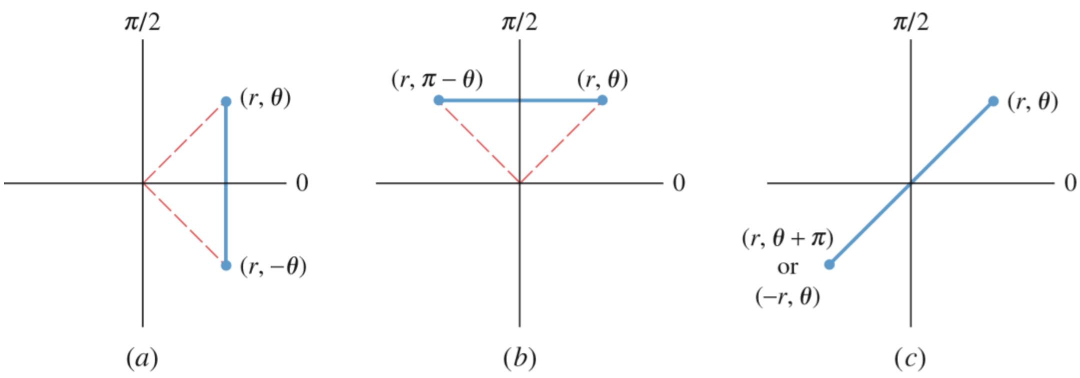
4.2.3 Symmetry Tests

Theorem (Symmetry Tests)

(a) A curve in polar coordinates is symmetric about the x-axis if replacing θ by $-\theta$ in its equation produces an equivalent equation (Figure a).

(b) A curve in polar coordinates is symmetric about the y-axis if replacing θ by $\pi-\theta$ in its equation produces an equivalent equation (Figure b).

(c) A curve in polar coordinates is symmetric about the origin if replacing θ by $\theta+\pi$, or replacing r by $-r$ in its equation produces an equivalent equation (Figure c).



Example 14

Use Theorem to confirm that the graph of $r = \cos 2\theta$ in Figure is symmetric about the x-axis and y-axis.

Solution. To test for symmetry about the x-axis, we replace θ by $-\theta$. This yields A

Note: A graph that is symmetric about both the x-axis and the y-axis is also symmetric about the origin. Use Theorem (c) to verify that the curve in Example 14 is symmetric about the origin.

$$r = \cos(-2\theta) = \cos 2\theta$$

Thus, replacing θ by $-\theta$ does not alter the equation.

To test for symmetry about the y-axis, we replace θ by $\pi - \theta$. This yields

$$r = \cos 2(\pi - \theta) = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos 2\theta$$

Thus, replacing θ by $\pi - \theta$ does not alter the equation.

Example 15

Sketch the graph of $r = a\,(1 - \cos\theta)$in polar coordinates, assuming $\boldsymbol{a}$ to be a positive constant.

Solution. Observe first that replacing θ by $-\theta$ does not alter the equation, so we know in advance that the graph is symmetric about the polar axis. Thus, if we graph the upper half of the curve, then we can obtain the lower half by reflection about the polar axis. As in our previous examples, we will first graph the equation in rectangular θ r-coordinates. This graph, which is shown in Figure a, can be obtained by rewriting the given equation as $r = a - a\cos\theta$, from which we see that the graph in rectangular θ r-coordinates can be obtained by first reflecting the graph of $r = a\cos\theta$ about the x-axis to obtain the graph of $r = -a\cos\theta$, and then translating that graph up a units to obtain the graph of $r = a - a\cos\theta$. Now we can see the following:

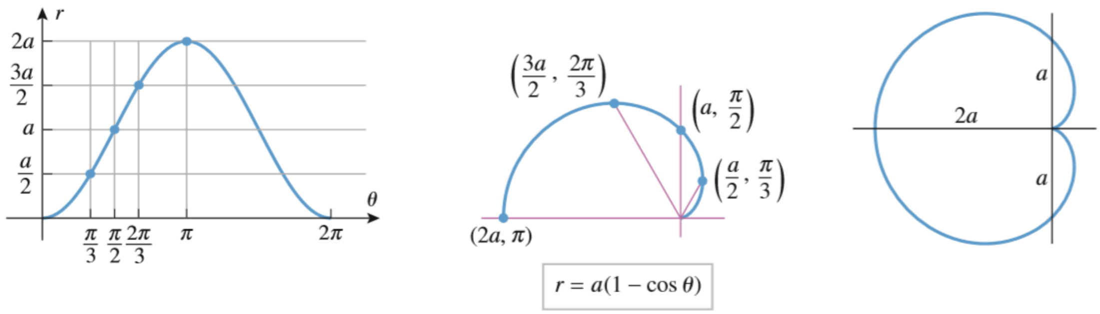
• As θ varies from 0 to π/3, r increases from 0 to a/2.

• As θ varies from π/3 to π/2, r increases from a/2 to a.

• As θ varies from $\pi/2$ to $2\pi/3$, r increases from a to $3a/2$.

• As θ varies from $2\pi/3$ to π, r increases from $3a/2$ to $2a$.

This produces the polar curve shown in Figure b. The rest of the curve can be obtained by continuing the preceding analysis from π to 2π or, as noted above, by reflecting the portion already graphed about the x-axis (Figure c). This heart-shaped curve is called a cardioid (from the Greek word kardia meaning "heart")



(a) (b) (c)

4.3 TANGENT LINES, ARC LENGTH, AND AREA FOR POLAR CURVES

4.3.1 Tangent Lines To Polar Curves

Our first objective in this section is to find a method for obtaining slopes of tangent lines to polar curves of the form $r = f(\theta)$in which r is a differentiable function of θ. We showed in the last section that a curve of this form can be expressed parametrically in terms of the parameter θ by substituting $f(\theta)$ for r in the equations $x = r\cos\theta$ and $y = r\sin\theta$. This yields

$$x = f(\theta)\cos\theta\,, \qquad y = f(\theta)\sin\theta$$

$$\frac{dx}{d\theta} = -f(\theta)\sin\theta + f'(\theta)\cos\theta = -r\sin\theta + \frac{dr}{d\theta}\cos\theta$$
$$\frac{dy}{d\theta} = f(\theta)\cos\theta + f'(\theta)\sin\theta = r\cos\theta + \frac{dr}{d\theta}\sin\theta$$

Thus, if $dx/d\theta$ and $dy/d\theta$ are continuous and if $dx/d\theta \neq 0$, then y is a differentiable function of x, and Formula (4) in with θ in place of t yields

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \sin\theta\dfrac{dr}{d\theta}}{-r\sin\theta + \cos\theta\dfrac{dr}{d\theta}}$$

Example 16

Find the slope of the tangent line to the circle $r = 4\cos\theta$ at the point where $\theta = \pi/4$.

Solution. From (2) with $r = 4\cos\theta$, so that $dr/d\theta = -4\sin\theta$, we obtain

$$\frac{dy}{dx} = \frac{4\cos^2\theta - 4\sin^2\theta}{-8\sin\theta\cos\theta} = -\frac{\cos^2\theta - \sin^2\theta}{2\sin\theta\cos\theta}$$

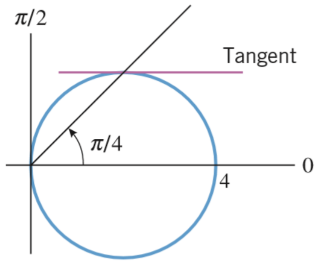
Using the double-angle formulas for sine and cosine,

$$\frac{dy}{dx} = -\frac{\cos 2\theta}{\sin 2\theta} = -\cot 2\theta$$

Thus, at the point where $\theta = \pi/4$ the slope of the tangent line is

$$m = \left.\frac{dy}{dx}\right|_{\theta=\pi/4} = -\cot\frac{\pi}{2} = 0$$

which implies that the circle has a horizontal tangent line at the point where $\theta = \pi/4$



$r = 4\cos\theta$

Example 17

Find the points on the cardioid $r = 1 - \cos\theta$ at which there is a horizontal tangent line, a vertical tangent line, or a singular point.

Solution. A horizontal tangent line will occur where $dy/d\theta = 0$ and $dx/d\theta \neq 0$, a vertical tangent line where $dy/d\theta \neq 0$ and $dx/d\theta = 0$, and a singular point where $dy/d\theta = 0$ and $dx/d\theta = 0$. We could find these derivatives from the formulas in (1). However, an alternative approach is to go back to basic principles and express the cardioid parametrically by substituting $r = 1 - \cos\theta$ in the conversion formulas $x = r\cos\theta$ and $y = r\sin\theta$. This yields

$$x = (1-\cos\theta)\cos\theta, \qquad y = (1-\cos\theta)\sin\theta \qquad (0 \le \theta \le 2\pi)$$

Differentiating these equations with respect to θ and then simplifying yields (verify)

$$\frac{dx}{d\theta} = \sin\theta(2\cos\theta - 1), \qquad \frac{dy}{d\theta} = (1-\cos\theta)(1+2\cos\theta)$$

Thus, $dx/d\theta = 0$ if $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$, and $dy/d\theta = 0$ if $\cos\theta = 1$ or $\cos\theta = -\frac{1}{2}$. We leave it for you to solve these equations and show that the solutions of $dx/d\theta = 0$ on the interval $0 \le \theta \le 2\pi$ are

$$\frac{dx}{d\theta} = 0: \quad \theta = 0,\ \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3},\ 2\pi$$

and the solutions of $dy/d\theta = 0$ on the interval $0 \le \theta \le 2\pi$ are

$$\frac{dy}{d\theta} = 0: \quad \theta = 0,\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ 2\pi$$

Thus, horizontal tangent lines occur at $\theta = 2\pi/3$ and $\theta = 4\pi/3$; vertical tangent lines occur at $\theta = \pi/3$, π, and $5\pi/3$; and singular points occur at $\theta = 0$ and $\theta = 2\pi$ (Figure 10.3.2). Note, however, that $r = 0$ at both singular points, so there is really only one singular point on the cardioid—the pole. ◄